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Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization

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Abstract
In Part I soliton solutions to the ABS list of multi-dimensionally consistent difference equations (except Q4) were derived using connection between the Q3 equation and the NQC equations, and then by reductions. In that work, the central role was played by a Cauchy matrix. In this work we use a different approach, and we derive the *N*-soliton solutions following Hirota's direct and constructive method. This leads to Casoratians and bilinear difference equations. We give here details for the H-series of equations and for Q1; the results for Q3 have been given earlier.

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1. Introduction

The analysis of integrability for discrete systems is now in active development. Discreteness introduces many complications in comparison with continuous integrability, mainly due to the lack of Leibniz rule for discrete derivatives. This also includes the fact that space-time itself has many discretizations.

The basic philosophy and definitions have been given in part I [1]; here we repeat only some essential ingredients. The underlying space is formed by a Cartesian square lattice and the dynamical equation is defined on an elementary square of this lattice. (There are other possible settings but even this case has not been fully analyzed.) As for the definition of integrability we choose 'Consistency-around-the cube' (CAC) which is further explained in section 2.1. With this choice of integrability many nice properties follow, including the existence of a Lax pair. Furthermore, with mild additional assumption one can classify the integrable models [2] and the 'ABS list' is surprisingly short.

In this paper we construct multi-soliton solutions for the H-series of models in the ABS-list, as well as for the Q1 model. Our approach is constructive for each model and is based on

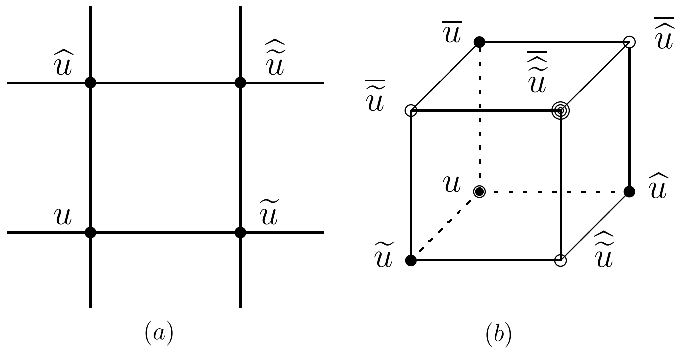

Figure 1. (*a*) The points on which the map is defined, and (*b*) the consistency cube.

the approach of Hirota, as explained in section 2.2. Some work on this problem already exists for the two highest members of the ABS-list, but the Q4 results in [3] are far from explicit and the result for Q3 [4] relied on a specific association with the NQC model, which has been further elaborated in part I.

We hope that the detailed analysis of the simpler models, for which everything can be made systematic and explicit, will provide further understanding on the soliton question and that it could be used for other models as well.

In the next section we will give the general background for our approach and then in subsequent sections we will go through the models H1, H2, H3 and Q1.

2. Generalities

2.1. Multidimensional consistency and the ABS list

As in part I we only consider quadrilateral lattice equations defined by a multi-linear relation on the values at the four corners of an elementary square of Cartesian lattice, see figure 1(*a*). Given a base point $u_{nm} = u$ we indicate the shifts in the n, m directions by a tilde or a hat as described in figure 1. Shifts in the opposite direction are denoted by under-tildes, under-hats etc, e.g., $u_{n-1,m,k} = \underset{\sim}{u}$.

Multidimensional consistency is an essential ingredient in the construction of soliton solutions, the new dimensions standing for parameters of the solitons. The consistency is defined as follows: we adjoin a third direction and construct an elementary cube as in figure 1(*b*), the shifts in the new third direction are denoted by a bar, e.g., $u_{n,m,k+1} = \overline{u}$. On the base square we have the equation $Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0$, and we now introduce the same equation with different variables on the sides and on the top, thus we will have altogether the following set of equations

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0, \qquad Q(\overline{u}, \overline{\widetilde{u}}, \overline{\widehat{u}}, \overline{\widehat{\widetilde{u}}}; p, q) = 0, \tag{2.1a}$$

$$Q(u, \widetilde{u}, \overline{u}, \overline{\widetilde{u}}; p, r) = 0, \qquad Q(\widehat{u}, \widehat{\widetilde{u}}, \overline{\widehat{u}}, \overline{\widehat{\widetilde{u}}}; p, r) = 0, \tag{2.1b}$$

$$Q(u, \widehat{u}, \overline{u}, \overline{\widehat{u}}; q, r) = 0, \qquad Q(\widetilde{u}, \widehat{\widetilde{u}}, \overline{\widetilde{u}}, \overline{\widehat{\widetilde{u}}}; q, r) = 0. \tag{2.1c}$$

Now considering figure 1(*b*) we can have initial values given at black circles $(u, \widetilde{u}, \widehat{u}, \overline{u})$, and in order to compute the remaining four values we have six equations. We can use the lhs equations in (2.1*a*), (2.1*b*), (2.1*c*) to compute $\widehat{\widetilde{u}}, \widetilde{\overline{u}}, \widehat{\overline{u}}$, respectively, and this leaves the three rhs equations from which we should get the same value for $\overline{\widehat{\widetilde{u}}}$, and this implies two consistency conditions.

In [2], a classification of quadrilateral lattice equations was performed following the above definition of integrability, with two additional requirements on the equations: symmetry and the so-called 'tetrahedron property', which means that the computed value for $\overline{\widehat{\widetilde{u}}}$ should only depend on $\widetilde{u}, \widehat{u}, \overline{u}$ and not on u. The ABS list is the following:

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+q-p=0, \tag{H1}$$

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+(q-p)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})+q^2-p^2=0, \tag{H2}$$

$$p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})+\delta(p^2-q^2)=0, \tag{H3}$$

$$p(u+\widehat{u})(\widetilde{u}+\widehat{\widetilde{u}})-q(u+\widetilde{u})(\widehat{u}+\widehat{\widetilde{u}})-\delta^2 pq(p-q)=0, \tag{A1}$$

$$(q^2-p^2)(u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}+1)+q(p^2-1)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-p(q^2-1)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})=0, \tag{A2}$$

$$p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})+\delta^2 pq(p-q)=0, \tag{Q1}$$

$$\begin{aligned} &p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})+pq(p-q)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})\\ &\qquad -pq(p-q)(p^2-pq+q^2)=0, \end{aligned} \tag{Q2}$$

$$\begin{aligned} &(q^2-p^2)(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u})+q(p^2-1)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-p(q^2-1)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})\\ &\qquad -\delta^2(p^2-q^2)(p^2-1)(q^2-1)/(4pq)=0, \end{aligned} \tag{Q3}$$

$$\begin{aligned} &(h(p)f(q)-h(q)f(p))[(u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}+1)f(p)f(q)-(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u})]\\ &\qquad +(f(p)^2f(q)^2-1)[(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})f(p)-(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})f(q)]=0, \end{aligned} \tag{Q4}$$

where $h^2 = f^4+\delta f^2+1$. This can be parameterized with Jacobi elliptic functions: $f(x) = k^{\frac{1}{4}}\,\mathrm{sn}(x,k), h(x) = \mathrm{sn}'(x,k), \delta = -(k+1/k)$. The simpler form of Q4 given above was actually discovered later in [5].

Here H1 is the discrete potential KdV equation and $H3_{\delta=0}$ is the discrete potential modified KdV equation. (For further historical remarks, see part I [1].)

The A-series is auxiliary in the sense that A1 goes to Q1 with $u \to (-1)^{n+m}u$ and A2 to $Q3_{\delta=0}$ by $u \to u^{(-1)^{n+m}}$. We will not discuss them here.

2.2. *Hirota's bilinear method*

Our approach to constructing soliton solutions is based on Hirota's bilinear method [6] introduced in 1971. The idea was to make a dependent variable transform into new variables, for which the soliton solution would be given by a polynomial of exponentials. In terms of the new dependent variables the dynamical equations turned out to be quadratic and derivatives appeared only in terms of Hirota's bilinear derivatives. These Hirota bilinear equations could then be solved perturbatively with a finite expansion. Hirota's idea has turned out to be quite successful and practically all integrable equations have been written in Hirota's bilinear form.

Soon after applying his method to continuous equations, Hirota discretized several soliton equations at the bilinear level by insisting on similar structure of multisoliton solutions [7].

In this paper we apply Hirota's original constructive method to the nonlinear equations H1–H3,Q1 of the ABS list. But what would then be the natural discrete generalization of the Hirota derivative? The essential property of an equation in Hirota's bilinear form seems to be its gauge invariance (cf [8] in the continuous case) and it has a natural discrete extension, leading us to the following definition:

Definition 1. *We say an equation is in Hirota bilinear (HB) form if it can be written as*

$$\sum_j c_j f_j\big(n+\nu_j^+, m+\mu_j^+\big) g_j\big(n+\nu_j^-, m+\mu_j^-\big) = 0, \tag{2.2}$$

where the index sums $\nu_j^+ + \nu_j^- = \nu^s$, $\mu_j^+ + \mu_j^- = \mu^s$ *do not depend on* j. *(The functions* f, g *may be the same.)*

Proposition 1. *Equations in HB form are gauge invariant, i.e., if functions* f_j, g_j *solve a set of equations in HB form, then so do the gauge transformed functions*

$$f_j'(n,m) = A^n B^m f_j(n,m), \qquad g_j'(n,m) = A^n B^m g_j(n,m). \tag{2.3}$$

Proof. We find

$$f_j'\big(n+\nu_j^+, m+\mu_j^+\big) g_j'\big(n+\nu_j^-, m+\mu_j^-\big)$$
$$= A^{2n+\nu_j^+ + \nu_j^-} B^{2m+\mu_j^+ + \mu_j^-} f_j\big(n+\nu_j^+, m+\mu_j^+\big) g_j\big(n+\nu_j^-, m+\mu_j^-\big)$$

but since the overall factor is the same in each term of the j-sum in (2.2) it can be taken out. □

For integrable equations in HB form there is a perturbative technique which leads to multi-soliton solutions, more or less algorithmically. This is described in the next section.

2.3. Constructing background solutions

First we have to construct the background or vacuum or seed solution, on top of which the soliton solutions are constructed. To do this we use the fixed-point idea [3] in which the consistent equations (2.1) are used with the assumption that $u = \overline{u}$. However, since some equations are invariant under some transformation $u \to T(u)$ it is actually sufficient that $\overline{u} = T(u)$. We only consider global invariances (that is, the transformation T is independent of n, m) and in order to keep the multi-linearity we assume that the transformation is linear fractional:

$$u_{nm} \to T(u_{nm}) := \frac{c_1 u_{nm} + c_2}{c_3 u_{nm} + c_4}, \qquad c_1 c_4 - c_2 c_3 \neq 0. \tag{2.4}$$

Depending on the equation there will be conditions on the parameters c_i. It is straightforward to find the invariances of the equations, which are given in table 1. Note that the special cases with $\delta = 0$ have a bigger invariance group. For Q4 there are various special cases depending on which limiting case of the elliptic parameterization one chooses.

For each invariance T of a given equation we then need to solve

$$Q(u, \widetilde{u}, T(u), T(\widetilde{u}); p, r) = 0, \tag{2.5a}$$
$$Q(u, \widehat{u}, T(u), T(\widehat{u}); q, r) = 0, \tag{2.5b}$$

in order to obtain the corresponding background solution (here r is a parameter of the solution). Such a solution then automatically solves (2.1a) by virtue of CAC.

It may be possible to construct still further solutions that can be called 'background solutions', but we will not consider them here.

Table 1. Invariances (2.4) of the equations in the ABS list.

Equation	$T(u)$	$T(u)$ when $\delta = 0$
H1	$u+c, -u+c$	NA
H2	u	NA
H3	$u, -u$	$cu, c/u$
A1	$u, -u$	$cu, c/u$
A2	$u, -u, 1/u, -1/u$	NA
Q1	$u+c, -u+c$	Full Möbius
Q2	u	NA
Q3	$u, -u$	$cu, c/u$
Q4	$u, -u, 1/u, -1/u$	Various

2.4. Constructing 1-soliton solutions

Once the background solution is obtained we construct a 1-soliton solution (1SS) using the CAC cube once more, with $\overline{u}$ now being the 1SS. This amounts, in fact, to a Bäcklund transformation (BT).

Thus the first task is to solve

$$Q(u, \widetilde{u}, \overline{u}, \widetilde{\overline{u}}; p, r) = 0, \tag{2.6a}$$

$$Q(u, \widehat{u}, \overline{u}, \widehat{\overline{u}}; q, r) = 0, \tag{2.6b}$$

where we take u to be the background solution u_0 and

$$\overline{u} = \overline{u}_0 + v, \tag{2.7}$$

where $\overline{u}_0$ is the bar shifted background solution. For a quadratic Q we can then write

$$Q(u, \widetilde{u}, \overline{u}, \widetilde{\overline{u}}; p, r) = Av\widetilde{v} + B\widetilde{v} + Cv + Q(u_0, \widetilde{u}_0, \overline{u}_0, \widetilde{\overline{u}}_0; p, r),$$

where A, B, C are some functions of u_0. Since the bar shift is structurally like any other shift, it follows from multidimensional consistency that the last term vanishes. We can then solve for $\widetilde{v}$ and $\widehat{v}$ from (2.6) in the form

$$\widetilde{v} = \frac{Ev}{v+F}, \qquad \widehat{v} = \frac{Gv}{v+H}, \tag{2.8}$$

where E, F, G, H may depend on n, m.

By introducing $v = g/f$ and $\Phi = (g, f)^T$ we can write (2.8) as a matrix equation

$$\Phi(n+1, m) = \mathcal{N}(n, m)\Phi(n, m), \qquad \Phi(n, m+1) = \mathcal{M}(n, m)\Phi(n, m), \tag{2.9}$$

where

$$\mathcal{N}(n, m) = \Lambda \begin{pmatrix} E & 0 \\ 1 & F \end{pmatrix}, \qquad \mathcal{M}(n, m) = \Lambda' \begin{pmatrix} G & 0 \\ 1 & H \end{pmatrix}. \tag{2.10}$$

The constants of separation Λ, Λ' are to be chosen so that $\widehat{\widetilde{\Phi}} = \widetilde{\widehat{\Phi}}$. In all cases studied in this paper it turns out that

$$\mathcal{N}(n, m) = \begin{pmatrix} S\frac{U_{n+1,m}}{U_{n,m}} & 0 \\ \frac{\sigma}{U_{n,m}} & \Delta \end{pmatrix}, \qquad \mathcal{M}(n, m) = \begin{pmatrix} T\frac{U_{n,m+1}}{U_{n,m}} & 0 \\ \frac{\tau}{U_{n,m}} & \Omega \end{pmatrix}, \tag{2.11}$$

where $S, T, \sigma, \tau, \Delta, \Omega$ are constants and U_{nm} some function of n, m (sometimes $U_{nm} = 1$). If we now introduce Ψ by

$$\Phi(n,m) = \begin{pmatrix} U_{n,m} & 0 \\ 0 & 1 \end{pmatrix} \Psi(n,m), \tag{2.12}$$

then for Ψ we have

$$\Psi(n+1,m) = \begin{pmatrix} S & 0 \\ \sigma & \Delta \end{pmatrix} \Psi(n,m), \qquad \Psi(n,m+1) = \begin{pmatrix} T & 0 \\ \tau & \Omega \end{pmatrix} \Psi(n,m). \tag{2.13}$$

These are compatible if

$$\frac{\sigma}{S-\Delta} = \frac{\tau}{T-\Omega}, \tag{2.14}$$

and in that case it is easy to derive

$$\Psi(n,m) = \begin{pmatrix} S^n T^m & 0 \\ \frac{\tau}{T-\Omega}(S^n T^m - \Delta^n \Omega^m) & \Delta^n \Omega^m \end{pmatrix} \Psi(0,0). \tag{2.15}$$

From this we can construct $\Phi(n, m)$ and then v: introduce

$$\rho_{n,m} = \left(\frac{S}{\Delta}\right)^n \left(\frac{T}{\Omega}\right)^m \rho_{0,0} \tag{2.16}$$

and then v can be written as

$$v_{n,m} = \frac{U_{n,m} \frac{v_{0,0}}{U_{0,0}} \rho_{n,m}/\rho_{0,0}}{1 + \frac{\tau}{T-\Omega} \frac{v_{0,0}}{U_{0,0}} (\rho_{n,m}/\rho_{0,0} - 1)}. \tag{2.17}$$

Now redefining the constant $\rho_{0,0}$ we can also write this as

$$v_{n,m} = U_{nm} \frac{T-\Omega}{\tau} \frac{\rho_{n,m}}{1+\rho_{n,m}}. \tag{2.18}$$

2.5. N-soliton solutions and Casoratians

Having a 1SS in the form mentioned above allows us to propose a change in dependent variables such that the original equation is given in terms of discrete equations in HB form. These equations will then be shown to have solutions given in Casorati determinant form, corresponding to the Wronskian form solutions for continuous HB equations. We will now discuss some generalities about the Casoratians.

In general the Casorati matrix is constructed as follows: given functions $\psi_i(n, m, l)$ we define the column vector

$$\psi(n,m,l) = (\psi_1(n,m,l), \psi_2(n,m,l), \ldots, \psi_N(n,m,l))^T, \tag{2.19a}$$

and then the generic $N \times N$ Casorati matrix is composed of such columns with different shifts l_i, with the determinant

$$C_{n,m}(\psi; \{l_i\}) = |\psi(n,m,l_1), \psi(n,m,l_2), \ldots, \psi(n,m,l_N)|. \tag{2.19b}$$

Two typical Casoratians that will be used later are

$$\begin{aligned} C^1_{n,m}(\psi) &:= |\psi(n,m,0), \psi(n,m,1), \ldots, \psi(n,m,N-1)| \\ &\equiv |0, 1, \ldots, N-1| \equiv |\widehat{N-1}|, \end{aligned} \tag{2.19c}$$

$$\begin{aligned} C^2_{n,m}(\psi) &:= |\psi(n,m,0), \ldots, \psi(n,m,N-2), \psi(n,m,N)| \\ &\equiv |0, 1, \ldots, N-2, N| \equiv |\widehat{N-2}, N|, \end{aligned} \tag{2.19d}$$

where we have also introduced the shorthand notation [10] in which only the shifts are given, and where furthermore a sequential change in the column index is indicated by a hat if it starts from 0: $|0, \ldots, M, \ldots| \equiv |\widehat{M}, \ldots|$. Later on we use another common notation in which a tilde indicates sequences starting from 1: $|\widetilde{M}| = |1, 2, \ldots, M|$. These are the standard notations used when dealing with Casoratians and since they only appear with Casorati determinants or matrices and always appear above numbers, there is no danger of confusing them with the tildes and hats used to denote shifts in the n or m indices.

In the Casoratians used in this work the entries ψ_i in the Nth order vector (2.19a) are given by

$$\psi_i(n, m, l) = \varrho_i^+(a + k_i)^n(b + k_i)^m(c + k_i)^l + \varrho_i^-(a - k_i)^n(b - k_i)^m(c - k_i)^l, \tag{2.20}$$

where $\varrho_i^\pm$ and k_i are parameters (and in some cases $c = 0$). In this form the shifts in l (bar-shifts) are not in any way special and therefore we can equally well define Casoratians where the shifts are in n (tilde-shifts) or in m (hat-shifts). For later use we indicate these three different shifts by the shift operators E^ν, $\nu = 1, 2, 3$, respectively, i.e.,

$$E^1\psi \equiv \widetilde{\psi}, \qquad E^2\psi \equiv \widehat{\psi}, \qquad E^3\psi \equiv \overline{\psi}.$$

Down shifts are denoted by E_ν, $\nu = 1, 2, 3$, respectively.

We can now define a Casoratian w.r.t. the three different kinds of shifts,

$$|\widehat{N-1}|_{[\nu]} = |\psi, E^\nu\psi, (E^\nu)^2\psi, \ldots, (E^\nu)^{N-1}\psi|, \qquad (\nu = 1, 2, 3). \tag{2.21}$$

Since

$$(\alpha_\mu - \alpha_\nu)\psi = (E^\mu - E^\nu)\psi, \qquad \mu, \nu = 1, 2, 3, \tag{2.22}$$

where

$$\alpha_1 \equiv a, \qquad \alpha_2 \equiv b, \qquad \alpha_3 \equiv c, \tag{2.23}$$

we have

$$(E^\nu)^j\psi = [E^\mu - (\alpha_\mu - \alpha_\nu)]^j\psi, \qquad \mu, \nu = 1, 2, 3,$$

and substituting this into (2.21) yields

$$|\widehat{N-1}|_{[1]} = |\widehat{N-1}|_{[2]} = |\widehat{N-1}|_{[3]}. \tag{2.24}$$

Using the shift relation (2.22), one can derive short expressions for shifted Casoratians $E_\mu|\widehat{N-1}|_{[\nu]} (\mu \neq \nu)$, and use them to prove solutions, see for example [11].

When the Casoratians are constructed using ψ of (2.20) the size on the matrix N indicates the number of solitons and the set $\{k_i\}_{i=1}^N$ provides the 'velocity' parameters of the solitons, while the parameters $\left\{\rho_i^+, \rho_i^-\right\}_1^N$ are related to the locations of the solitons (by gauge invariance only their ratio is significant).

We shall here mention that the bilinear equations we get in the paper are more or less similar to the Hirota–Miwa equation [12, 13]

$$a(b - c)\tau_{n,m,l+1}\tau_{n+1,m+1,l} + b(c - a)\tau_{n,m+1,l}\tau_{n+1,m,l+1} + c(a - b)\tau_{n+1,m,l}\tau_{n,m+1,l+1} = 0, \tag{2.25}$$

(cf (5.17, 5.20)), or belong to the bilinear equations which were derived by imposing the transformation

$$x_j = \sum_{i=1}^{\infty} l_j \frac{a_i^j}{j} \tag{2.26}$$

on Sato's bilinear identity. The above transformation, referred to as Miwa transformation [13, 14], provides a connection between continuous coordinates $\{x_j\}$ and discrete ones $\{l_j\}$, and transforms the basic continuous plane wave factor

$$\exp\left[-\sum_{jx=1}^{\infty} x_j p_i + \sum_{j=1}^{\infty} x_j q_i\right] \tag{2.27}$$

into the discrete one

$$\prod_{j=1}^{\infty}\left(\frac{1-a_j p_i}{1-a_j q_i}\right)^{l_j}. \tag{2.28}$$

The discrete exponential function (2.28) corresponding to the plane wave factor ρ_i (I-2.2), plays a central role in the discrete τ function in Hirota's exponential-polynomial form, while in Casoratians its counterpart has the form (2.20).

In most cases a bilinear equation that can be solved by an $N \times N$ Casoratian (or Casoratians) is reduced to a Laplace expansion of a $2N \times 2N$ determinant with zero value. For later convenience we give the following lemmas.

Lemma 1 [10]**.**

$$\sum_{j=1}^{N} |\mathbf{a}_1, \ldots, \mathbf{a}_{j-1}, \mathbf{b}\mathbf{a}_j, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_N| = \left(\sum_{j=1}^{N} b_j\right) |\mathbf{a}_1, \ldots, \mathbf{a}_N|, \tag{2.29}$$

where $\mathbf{a}_j = (a_{1j}, \ldots, a_{Nj})^T$ *and* $\mathbf{b} = (b_1, \ldots, b_N)^T$ *are N-order column vectors, and* $\mathbf{b}\mathbf{a}_j$ *stands for* $(b_1 a_{1j}, \ldots, b_N a_{Nj})^T$.

Lemma 2 [10]**.** *Suppose that* $\mathbf{B}$ *is an* $N \times (N-2)$ *matrix and* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *are N-order column vectors, then*

$$|\mathbf{B}, \mathbf{a}, \mathbf{b}||\mathbf{B}, \mathbf{c}, \mathbf{d}| - |\mathbf{B}, \mathbf{a}, \mathbf{c}||\mathbf{B}, \mathbf{b}, \mathbf{d}| + |\mathbf{B}, \mathbf{a}, \mathbf{d}||\mathbf{B}, \mathbf{b}, \mathbf{c}| = 0. \tag{2.30}$$

In fact, the lhs of (2.30) is just the Laplace expansion of the following $2N \times 2N$ determinant,

$$\frac{1}{2}\begin{vmatrix} \mathbf{B} & \mathbf{0} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{0} & \mathbf{B} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{vmatrix} \equiv 0.$$

3. H1

3.1. Background solution and 1-soliton solution

3.1.1. The background solution. The H1 equation is given by

$$\mathrm{H1} \equiv (u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) - (p - q) = 0. \tag{3.1}$$

Using the fixed point idea with transformation $\overline{u} = T(u) = u + c$ we get the side-equations of the CAC-cube in the form

$$(\widetilde{u} - u)^2 = r + c^2 - p, \qquad (\widehat{u} - u)^2 = r + c^2 - q.$$

For convenience we absorb r into c^2, and reparameterize $(p, q) \to (a, b)$ by

$$p = c^2 - a^2, \qquad q = c^2 - b^2 \tag{3.2}$$

and then the above equations factorize as

$$(\widetilde{u} - u - a)(\widetilde{u} - u + a) = 0, \qquad (\widehat{u} - u - b)(\widehat{u} - u + b) = 0. \tag{3.3}$$

Since the factor that vanishes may depend on n, m we actually have to solve

$$\widetilde{u} - u = (-1)^{\theta} a, \qquad \widehat{u} - u = (-1)^{\sigma} b, \tag{3.4}$$

where $\theta, \sigma \in \mathbb{Z}$ may depend on n, m. Furthermore, since the value of the exponent is only relevant modulo 2 and $n^2 \equiv n \bmod 2$, the exponents must be linear combinations of $1, n, m, nm$ with coefficients 0 or 1.

The integrability condition for (3.4) leads to

$$b[(-1)^{\sigma} - (-1)^{\widetilde{\sigma}}] = a[(-1)^{\theta} - (-1)^{\widehat{\theta}}],$$

and since constants a, b are independent this just means that $\sigma = \sigma(m), \theta = \theta(n)$. Since these are defined modulo 2 we can have $\theta = t_1 n + t_0, \sigma = s_1 m + s_0$ where $t_i, s_i \in \{0, 1\}$. Furthermore, since a, b were only defined up to sign we may take $t_0 = s_0 = 0$.

There are now two essentially different cases for each exponent, either $\theta(n) = 0$ or n, and $\sigma(m) = 0$ or m. The choice $\theta = 0$ leads to a solution $u = an + \cdots$ while $\theta = n$ leads to $u = -\frac{1}{2}(-1)^n a + \cdots$. These solutions can be combined, and thus we get the following set of possibilities for $u^{0\mathrm{SS}}$:

$$an + bm + \gamma, \tag{3.5a}$$

$$\tfrac{1}{2}(-1)^n a + bm + \gamma, \tag{3.5b}$$

$$an + \tfrac{1}{2}(-1)^m b + \gamma, \tag{3.5c}$$

$$\tfrac{1}{2}(-1)^n a + \tfrac{1}{2}(-1)^m b + \gamma. \tag{3.5d}$$

We could also try to get a background solution using the fixed point idea with the other transformation mentioned for H1 in table 1, namely $\overline{u} = T(u) = -u + c$, but it produces the same set: with a reparameterization

$$p = r + \alpha^2, \qquad q = r + \beta^2, \qquad u = y + \tfrac{1}{2}c. \tag{3.6}$$

the side equations become

$$(\widetilde{y} - y - \alpha)(\widetilde{y} - y + \alpha) = 0, \qquad (\widehat{y} - y - \beta)(\widehat{y} - y + \beta) = 0, \tag{3.7}$$

which is as in (3.3). The constant c can be absorbed into the constant γ and the free parameter r redefined when comparing (3.2), (3.6), and thus the background solutions are as in (3.5).

It seems that only (3.5a) leads to a soliton-type solution so in the following we only consider it.

3.1.2. Constructing 1SS using a BT. The BT generating 1SS for H1 is

$$(u - \widetilde{\overline{u}})(\widetilde{u} - \overline{u}) = p - \varkappa, \tag{3.8a}$$

$$(u - \widehat{\overline{u}})(\overline{u} - \widehat{u}) = \varkappa - q. \tag{3.8b}$$

Here u is the seed solution (3.5a), $\varkappa$ is the parameter in the bar-direction, and we search for a new solution $\overline{u}$ of the form

$$\overline{u} = \overline{u}_0 + v, \tag{3.9}$$

where $\overline{u}_0$ is the bar-shifted background solution (3.5*a*). As explained in section 2.4 the bar shift must have the same form as any other shift and thus if we write

$$\overline{u}_0 = an + bm + k + \gamma, \tag{3.10}$$

then, in analogue to (3.2), the parameters k and $\varkappa$ must be related by

$$\varkappa = c^2 - k^2. \tag{3.11}$$

Then substituting (3.5*a*), (3.9) and (3.10) into (3.8) yields the desired form

$$\widetilde{v} = \frac{Ev}{v+F}, \qquad \widehat{v} = \frac{Gv}{v+H}, \tag{3.12}$$

where

$$E = -(a+k), \qquad F = -(a-k), \qquad G = -(b+k), \qquad H = -(b-k).$$

Now matrices $\mathcal{N}$, $\mathcal{M}$ defined in (2.10) are n, m independent and since $E - F = G - H = -2k$ they commute and we can take $\Lambda = \Lambda' = 1$. Then, following the method presented in section 2.4, we find

$$\Phi(n,m) = \begin{pmatrix} E^n G^m & 0 \\ \frac{E^n G^m - F^n H^m}{-2k} & F^n H^m \end{pmatrix} \Phi(0,0), \tag{3.13}$$

and if we let

$$\rho_{n,m} = \left(\frac{E}{F}\right)^n \left(\frac{G}{H}\right)^m \rho_{0,0} = \left(\frac{a+k}{a-k}\right)^n \left(\frac{b+k}{b-k}\right)^m \rho_{0,0}, \tag{3.14}$$

then we obtain

$$v_{n,m} = \frac{-2k\rho_{n,m}}{1+\rho_{n,m}}. \tag{3.15}$$

Finally we obtain the 1SS for H1:

$$u_{n,m}^{1\mathrm{SS}} = (an + bm + \gamma) + k + \frac{-2k\rho_{n,m}}{1+\rho_{n,m}}. \tag{3.16}$$

3.2. *Multi-soliton solutions*

In an explicit form the 1SS given above ((3.16) with (3.14)) is

$$\begin{aligned} u_{n,m}^{1\mathrm{SS}} &= an + bm + \gamma + \frac{k(1-\rho_{n,m})}{1+\rho_{n,m}} \\ &= an + bm + \gamma - \frac{k[\varrho^+(a+k)^n(b+k)^m - \varrho^-(a-k)^n(b-k)^m]}{\varrho^+(a+k)^n(b+k)^m + \varrho^-(a-k)^n(b-k)^m}, \end{aligned} \tag{3.17}$$

where we have separated $\rho_{0,0}$ to ϱ^-/ϱ^+.

The above suggests that the basic ingredient in constructing an NSS is ψ of (2.20) with $c = 0$, i.e.,

$$\psi_i(n,m,l;k_i) = \varrho_i^+ k_i^l (a+k_i)^n (b+k_i)^m + \varrho_i^- (-k_i)^l (a-k_i)^n (b-k_i)^m. \tag{3.18}$$

Here the index l is needed as a column index in the Casorati matrix.

The form of the 1SS (3.17) suggests a generalization to NSS, and after checking explicitly the 2SS and 3SS, we arrive at the following proposition:

Proposition 2. *N-soliton solution for H1 is given by*

$$u_{n,m}^{\mathrm{NSS}} = an + bm + \gamma - \frac{g}{f}, \tag{3.19}$$

where $f = |\widehat{N-1}|_{[3]}$, $g = |\widehat{N-2}, N|_{[3]}$, *with* ψ *given by (3.18).*

Note that this solution is the same as (5.29) of part I with $A = 0$. Indeed using (I-2.1, 2.3, 2.8c) one finds that

$$S^{(0,0)} - \sum k_i = \frac{g}{f},$$

and the parameters $\varrho_i^{\pm}$ in (3.18) and c_i in (I-2.3) are related by

$$\frac{\varrho_i^+}{\varrho_i^-} = \frac{c_i}{2k_i} \prod_{j \neq i} \frac{k_j - k_i}{k_j + k_i}. \tag{3.20}$$

However, solution (3.19) contains more freedom in the form of the additional parameter c, cf (3.2) and (I-5.1d).

3.2.1. The bilinear form of H1. In order to prove proposition 2 we write (3.1) in bilinear form. First we introduce a dependent variable transformation

$$u_{n,m}^{\mathrm{NSS}} = an + bm + \gamma - \frac{g_{n,m}}{f_{n,m}}. \tag{3.21}$$

When this is substituted into (3.1) we get a rational expression quartic in f, g. In order to split this expression into bilinear equations we note that if $f = |\widehat{N-1}|_{[3]}$, $g = |\widehat{N-2}, N|_{[3]}$, then for $N = 2$ one can quickly scan for possible equations solved by them from the set

$$a_0 g \widehat{\widetilde{f}} + a_1 \widehat{g}\widetilde{f} + a_2 \widetilde{g}\widehat{f} + a_3 \widehat{\widetilde{g}} f + a_4 f \widehat{\widetilde{f}} + a_5 \widehat{f}\widetilde{f} + a_6 g \widehat{\widetilde{g}} + a_7 \widehat{g}\widetilde{g} = 0,$$

and find the bilinear equations

$$\mathcal{H}_1 \equiv \widehat{g}\widetilde{f} - \widetilde{g}\widehat{f} + (a-b)(\widehat{f}\widetilde{f} - f\widehat{\widetilde{f}}) = 0, \tag{3.22a}$$

$$\mathcal{H}_2 \equiv g\widehat{\widetilde{f}} - \widehat{\widetilde{g}} f + (a+b)(f\widehat{\widetilde{f}} - \widehat{f}\widetilde{f}) = 0. \tag{3.22b}$$

After this it is easy to show that

$$\begin{aligned} \mathrm{H1} &\equiv (u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) - p + q \\ &= -[\mathcal{H}_1 + (a-b) f\widehat{\widetilde{f}}][\mathcal{H}_2 + (a+b)\widehat{f}\widetilde{f}]/(f\widehat{f}\widetilde{f}\widehat{\widetilde{f}}) + (a^2 - b^2), \end{aligned}$$

and thus the pair (3.22) can be taken as the bilinear form of (3.1).

The proof that f, g given in proposition 2 solve equations (3.22) has been given in [15].

4. H2

4.1. Background solution

The equation H2 is given by

$$\mathrm{H2} \equiv (u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) - (p - q)(u + \widetilde{u} + \widehat{u} + \widehat{\widetilde{u}} + p + q) = 0. \tag{4.1}$$

After reparameterization

$$p = r - a^2, \qquad q = r - b^2, \tag{4.2}$$

the equations on sides become

$$\begin{aligned} (\widetilde{u} - u)^2 - 2a^2(\widetilde{u} + u) + a^2(a^2 - 2r) &= 0, \\ (\widehat{u} - u)^2 - 2b^2(\widehat{u} + u) + b^2(b^2 - 2r) &= 0. \end{aligned} \tag{4.3}$$

After the further substitution

$$u = y^2 - \tfrac{1}{2} r \tag{4.4}$$

equations (4.3) factorize as

$$(\widetilde{y}+y+a)(\widetilde{y}+y-a)(\widetilde{y}-y+a)(\widetilde{y}-y-a)=0, \tag{4.5a}$$

$$(\widehat{y}+y+b)(\widehat{y}+y-b)(\widehat{y}-y+b)(\widehat{y}-y-b)=0. \tag{4.5b}$$

Thus we have to solve

$$\widetilde{y}-(-1)^{\sigma}y=(-1)^{\theta}a, \qquad \widehat{y}-(-1)^{\rho}y=(-1)^{\phi}b, \tag{4.6}$$

where the exponents $\sigma,\theta,\rho,\phi\in\mathbb{Z}$ are again some linear combination of $1,n,m,nm$ with coefficients 0 or 1. Equations (4.6) are satisfied if

$$\rho=ns_2+r_1, \qquad \sigma=ms_2+s_1,$$

$$\theta=m\rho+\sigma+n(t_n+s_1)+t_1, \qquad \phi=n\sigma+\rho+m(p_m+r_1)+p_1,$$

but most of the freedom is superfluous: since the sign of y was left undetermined in (4.4) we can redefine

$$y_{n,m}\to(-1)^{nms_2+ns_1+mr_1}y_{n,m}$$

and then the equations simplify to

$$\widetilde{y}-y=(-1)^{nt_n+t_1}a, \qquad \widehat{y}-y=(-1)^{mp_m+p_1}b,$$

already analyzed after (3.4). Thus we find the possible background solutions $u^{0SS}_{n,m}$ of the form

$$[an+bm+\gamma]^2-\tfrac{1}{2}r, \tag{4.7a}$$

$$\left[\tfrac{1}{2}(-1)^n a+mb+\gamma\right]^2-\tfrac{1}{2}r, \tag{4.7b}$$

$$\left[na+\tfrac{1}{2}(-1)^m b+\gamma\right]^2-\tfrac{1}{2}r, \tag{4.7c}$$

$$\left[\tfrac{1}{2}(-1)^n a+\tfrac{1}{2}(-1)^m b+\gamma\right]^2-\tfrac{1}{2}r. \tag{4.7d}$$

4.2. 1-soliton solution

Note that H2 is invariant under simultaneous translation of p,q by t and u by $-\frac{1}{2}t$. We use this freedom to eliminate r in order to simplify the presentation.

Now we take (4.7a) as the seed solution to construct $u^{1SS}_{n,m}$ for H2 through its Bäcklund transformation

$$(u-\widetilde{\overline{u}})(\widetilde{u}-\overline{u})=(p-\varkappa)(u+\widetilde{u}+\overline{u}+\widetilde{\overline{u}}+p+\varkappa), \tag{4.8a}$$

$$(u-\widehat{\overline{u}})(\overline{u}-\widehat{u})=(\varkappa-q)(u+\overline{u}+\widehat{u}+\widehat{\overline{u}}+\varkappa+q), \tag{4.8b}$$

where u is the seed solution (4.7a) and we search for a new solution $\overline{u}$ of the form

$$\overline{u}=\overline{u}_0+v, \tag{4.9}$$

where $\overline{u}_0$ is the bar-shifted background solution (4.7a):

$$\overline{u}_0=(an+bm+k+\gamma)^2, \tag{4.10}$$

with

$$\varkappa=k^2. \tag{4.11}$$

Substituting these into (4.8) yields (2.10) with

$$E = -2(k+a)[(n+1)a + mb + \gamma], \qquad F = -2(k-a)[na + mb + \gamma],$$
$$G = -2(k+b)[na + (m+1)b + \gamma], \qquad H = -2(k-b)[na + mb + \gamma]$$

and the corresponding matrices $\mathcal{N}$, $\mathcal{M}$ are compatible if we take

$$\Lambda = \Lambda' = -1/(2U_{nm}), \qquad U_{n,m} := an + bm + \gamma. \tag{4.12}$$

With this U we obtain (2.11) with

$$S = a + k, \qquad \Delta = a - k, \qquad T = b + k,$$
$$\Omega = b - k, \qquad \sigma = \tau = -1/2. \tag{4.13}$$

Then defining

$$\rho_{n,m} = \left(\frac{S}{\Delta}\right)^n \left(\frac{T}{\Omega}\right)^m \rho_{0,0} = \left(\frac{a+k}{a-k}\right)^n \left(\frac{b+k}{b-k}\right)^m \rho_{0,0}, \tag{4.14}$$

we find

$$v_{n,m} = \frac{-4kU_{n,m}\rho_{n,m}}{1 + \rho_{n,m}}, \tag{4.15}$$

and finally we obtain the 1-soliton for H2 in the form

$$u^{1SS}_{n,m} = U^2_{n,m} + 2kU_{n,m}\frac{1 - \rho_{n,m}}{1 + \rho_{n,m}} + k^2. \tag{4.16}$$

4.3. Multi-soliton solution

Motivated by the structure of 1SS (4.16) (cf (3.17)), and after checking 2- and 3-soliton solutions we propose the following Casoratian expression for the NSS:

$$u^{NSS}_{n,m} = U^2_{n,m} - 2U_{n,m}\frac{|\widehat{N-2}, N|_{[3]}}{|\widehat{N-1}|_{[3]}} + \frac{|\widehat{N-3}, N-1, N|_{[3]} + |\widehat{N-2}, N+1|_{[3]}}{|\widehat{N-1}|_{[3]}}, \tag{4.17}$$

where the matrix entries are as for H1 (3.18). In order to prove this we derive a bilinear form of H2. We propose

$$u^{NSS}_{n,m} = U^2_{n,m} - 2U_{n,m}\frac{g}{f} + \frac{h+s}{f}, \tag{4.18}$$

where f, h and s should satisfy

$$h - s = \gamma f, \tag{4.19}$$

where γ is some constant. Indeed, if

$$f = |\widehat{N-1}|_{[3]}, \qquad g = |\widehat{N-2}, N|_{[3]},$$
$$s = |\widehat{N-3}, N-1, N|_{[3]}, \qquad h = |\widehat{N-2}, N+1|_{[3]}, \tag{4.20}$$

then noting that $(E^3)^2\psi_i(n, m, l) = \overline{\overline{\psi}}_i(n, m, l) = \psi_i(n, m, l+2) = k_i^2\psi_i(n, m, l)$ and using lemma 1 we have $\left(\sum_{i=1}^N k_i^2\right) f = h - s$, i.e. $\gamma = \sum_{i=1}^N k_i^2$.

The solution (4.18) is the same as (I-5.26) with $A = 0$ and

$$S^{(0,0)} = \sum_j k_j + \frac{g}{f}, \qquad 2S^{(0,1)} - 2\left(\sum_j k_j\right) S^{(0,0)} + \left(\sum_j k_j\right)^2 = \frac{h+s}{f},$$

which can be found by using (I-2.1, 2.2, 2.3, 2.8c, 5.23).

Under condition (4.19), H2 can be represented through the following bilinear system,

$$\mathcal{H}_1 \equiv \widehat{g}\widetilde{f} - \widetilde{g}\widehat{f} + (a-b)(\widehat{f}\widetilde{f} - f\widehat{\widetilde{f}}) = 0, \qquad (4.21a)$$

$$\mathcal{H}_2 \equiv g\widehat{\widetilde{f}} - \widehat{\widetilde{g}}f + (a+b)(f\widehat{\widetilde{f}} - \widehat{f}\widetilde{f}) = 0, \qquad (4.21b)$$

$$\mathcal{H}_3 \equiv -(a+b)\widehat{f}\widetilde{g} + a\widehat{\widetilde{f}}g + bf\widehat{\widetilde{g}} + \widehat{\widetilde{f}}h - f\widehat{\widetilde{h}} = 0, \qquad (4.21c)$$

$$\mathcal{H}_4 \equiv -(a-b)f\widehat{\widetilde{g}} + a\widetilde{f}\widehat{g} - b\widehat{f}\widetilde{g} + \widetilde{f}\widehat{h} - \widehat{f}\widetilde{h} = 0, \qquad (4.21d)$$

$$\mathcal{H}_5 \equiv b(\widehat{f}g - f\widehat{g}) + f\widehat{h} + \widehat{f}s - g\widehat{g} = 0, \qquad (4.21e)$$

in which $\mathcal{H}_1$ and $\mathcal{H}_2$ already appeared in (3.22). In terms of the above bilinear equations H2 can be given as

$$\mathrm{H2} = \sum_{i=1}^{5} \mathcal{H}_i P_i, \qquad (4.22)$$

with

$$P_1 = -4(a+b)[(\widetilde{U}\widehat{\widetilde{U}} - a^2 + b^2)\widetilde{f}\widehat{f} - \widehat{\widetilde{U}}\widehat{f}\widetilde{g} - (a+b)f\widehat{\widetilde{g}}],$$

$$P_2 = -4[(a-b)(\widehat{U}\widehat{\widetilde{U}} - a^2 + b^2)\widetilde{f}\widehat{f} + (\widetilde{U}\widehat{\widetilde{U}} - a^2 + b^2)\widetilde{f}\widehat{g} - \widetilde{U}\widehat{\widetilde{U}}\widehat{f}\widetilde{g} - (a-b)\widetilde{U}f\widehat{\widetilde{g}}],$$

$$P_3 = 4[(a-b)U\widetilde{f}\widehat{f} + \widehat{U}\widetilde{f}\widehat{g} - \widetilde{U}\widehat{f}\widetilde{g} - \widetilde{f}\widehat{h} + \widehat{f}\widetilde{h}],$$

$$P_4 = 4[(a+b)(\widehat{U}f\widehat{\widetilde{f}} - \widehat{f}\widetilde{g}) + \widetilde{U}(\widehat{\widetilde{f}}g - f\widehat{\widetilde{g}})],$$

$$P_5 = 4(a^2 - b^2)\widetilde{f}\widehat{\widetilde{f}},$$

where U was defined in (4.12). It remains to prove the following proposition.

Proposition 3. *The Casoratian type determinants f, g, h and s given in (4.20) with entries given by (3.18) solve the set of bilinear equations (4.21).*

Proof. Among (4.21) $\mathcal{H}_1$ and $\mathcal{H}_2$ already appeared in section 3.2. Next we prove (4.21c) in the following form

$$-(a+b)\widehat{\utilde{f}}g + \widehat{f}(\utilde{h} + a\utilde{g}) - \utilde{f}(\widehat{h} - b\widehat{g}) = 0, \qquad (4.23)$$

which is a down-tilde-shifted version of the original one. Since $\psi_i(n, m, l)$ given by (3.18) is just (A.1) with $c = 0$, we use the formulae given in appendix A with $c \equiv 0$ and $\kappa = 3$. In (4.23) $g = |\widehat{N-2}, N|_{[3]}$, and for $\widehat{\utilde{f}}$, $\widehat{f}$, $\utilde{h} + a\utilde{g}$, $\utilde{f}$ and $\widehat{h} - b\widehat{g}$ we use (A.6l) with $\mu = 2$ and $\nu = 1$, (A.6c) with $\mu = 2$, (A.8a) with $\mu = 1$, (A.6a) with $\mu = 1$ and (A.8b) with $\mu = 2$, respectively. Then we have

$$\begin{aligned}
&\frac{1}{|\Omega_2|}a^{N-2}b^{N-2}[-(a+b)\widehat{\utilde{f}}g + \widehat{f}(\utilde{h} + a\utilde{g}) - \utilde{f}(\widehat{h} - b\widehat{g})] \\
&\quad = -|\widehat{N-2}, N|_{[3]}|\widehat{N-3}, \utilde{\psi}(N-2), \mathring{E}^2\psi(N-2)|_{[3]} \\
&\qquad - |\widehat{N-2}, \mathring{E}^2\psi(N-2)|_{[3]}|\widehat{N-3}, N, \utilde{\psi}(N-2)|_{[3]} \\
&\qquad + |\widehat{N-2}, \utilde{\psi}(N-2)|_{[3]}|\widehat{N-3}, N, \mathring{E}^2\psi(N-2)|_{[3]} \\
&\quad = 0,
\end{aligned}$$

where we have made use of lemma 2 in which $\mathbf{B} = (\widehat{N-3})$, $\mathbf{a} = \psi(N-2)$, $\mathbf{b} = \psi(N)$, $\mathbf{c} = \utilde{\psi}(N-2)$ and $\mathbf{d} = \mathring{E}^2\psi(N-2)$

(4.21d) can be proved similarly after a down-tilde-hat-shift.

Next we prove (4.21*e*), which can be written as

$$f(\widehat{h}-b\widehat{g})-g(\widehat{g}-b\widehat{f})+\widehat{f}s=0, \tag{4.24}$$

where $f=|\widehat{N-1}|_{[3]}$, $g=|\widehat{N-2},N|_{[3]}$, $s=|\widehat{N-3},N-1,N|_{[3]}$, and $\widehat{h}-b\widehat{g}$, $\widehat{g}-b\widehat{f}$ and $\widehat{f}$ will be provided by (A.8*b*) with $\mu=2$, (A.7*b*) with $\mu=2$ and (A.6*c*) with $\mu=2$, respectively. Then we have

$$\begin{aligned}\frac{1}{|\Omega_2|}b^{N-2}[f(\widehat{h}-b\widehat{g})-g(\widehat{g}-b\widehat{f})+\widehat{f}s] &= |\widehat{N-1}|_{[3]}|\widehat{N-3},N,\mathring{E}^2\psi(N-2)|_{[3]}\\ &\quad -|\widehat{N-2},N|_{[3]}|\widehat{N-3},N-1,\mathring{E}^2\psi(N-2)|_{[3]}\\ &\quad +|\widehat{N-2},\mathring{E}^2\psi(N-2)|_{[3]}|\widehat{N-3},N-1,N|_{[3]}\\ &=0,\end{aligned}$$

where use has been made of lemma 2 with $\mathbf{B}=(\widehat{N-3})$, $\mathbf{a}=\psi(N-2)$, $\mathbf{b}=\psi(N-1)$, $\mathbf{c}=\psi(N)$ and $\mathbf{d}=\mathring{E}^2\psi(N-2)$. □

5. H3

5.1. Background solution

H3 is given by

$$\text{H3}\equiv p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-\delta(q^2-p^2)=0. \tag{5.1}$$

The side equations for $T(x)=x$ then read

$$r(u^2+\widetilde{u}^2)-2pu\widetilde{u}=\delta(p^2-r^2),\qquad r(u^2+\widehat{u}^2)-2qu\widehat{u}=\delta(q^2-r^2). \tag{5.2}$$

In this case we reparameterize

$$\begin{aligned} &p=r\cosh(\alpha'), && q=r\cosh(\beta'),\\ &u_{nm}=A\,\mathrm{e}^{y_{nm}}+B\,\mathrm{e}^{-y_{nm}}, && AB=-\tfrac{1}{4}r\delta \end{aligned} \tag{5.3}$$

and then equations (5.2) factorize as

$$(\mathrm{e}^{\widetilde{y}-y+\alpha'}-1)(\mathrm{e}^{\widetilde{y}-y-\alpha'}-1)\left(\mathrm{e}^{\widetilde{y}+y+\alpha'-\ln\frac{B}{A}}-1\right)\left(\mathrm{e}^{\widetilde{y}+y-\alpha'-\ln\frac{B}{A}}-1\right)=0, \tag{5.4a}$$

$$(\mathrm{e}^{\widehat{y}-y+\beta'}-1)(\mathrm{e}^{\widehat{y}-y-\beta'}-1)\left(\mathrm{e}^{\widehat{y}+y+\beta'-\ln\frac{B}{A}}-1\right)\left(\mathrm{e}^{\widehat{y}+y-\beta'-\ln\frac{B}{A}}-1\right)=0. \tag{5.4b}$$

Since we only consider real u the various possibilities can be represented as in (4.6). The analysis is then the same, especially since also here the sign of y is undetermined in (5.3). Thus the solution for y is as in (3.5) and for u we have

$$\begin{aligned} u^{0SS}&=A\,\mathrm{e}^{y_{nm}}+B\,\mathrm{e}^{-y_{nm}}=A\,\mathrm{e}^{\alpha' n+\beta' m+\gamma}+B\,\mathrm{e}^{-\alpha' n-\beta' m-\gamma}\\ &=A\alpha^n\beta^m+B\alpha^{-n}\beta^{-m},\qquad AB=-\tfrac{1}{4}r\delta, \end{aligned} \tag{5.5}$$

where $\alpha=\mathrm{e}^{\alpha'}$, $\beta=\mathrm{e}^{\beta'}$.

In the case of $T(x)=-x$, one can find that the side equations are just as for $T(x)=x$ except that $r\to -r$. Since r is a free parameter this adds nothing new.

5.2. 1-soliton solution for H3

Now we take (5.5) as the seed solution to construct $u^{1SS}_{n,m}$ for H3 through its Bäcklund transformation

$$p(u\widetilde{u}+\overline{u}\widetilde{\overline{u}})-\varkappa(u\overline{u}+\widetilde{u}\widetilde{\overline{u}})=\delta(\varkappa^2-p^2), \tag{5.6a}$$

$$\varkappa(u\overline{u}+\widehat{u}\widehat{\overline{u}})-q(u\widehat{u}+\overline{u}\widehat{\overline{u}})=\delta(q^2-\varkappa^2), \tag{5.6b}$$

where u is the seed solution (5.5), and we search for the 1SS $\overline{u}$ of the form

$$\overline{u}=\overline{u}_0+v, \tag{5.7}$$

where $\overline{u}_0$ is the bar-shifted background solution (5.5):

$$\overline{u}_\theta=A\alpha^n\beta^m\kappa+B\alpha^{-n}\beta^{-m}\kappa^{-1}, \tag{5.8}$$

with $\varkappa$ and κ related by

$$\varkappa=r\frac{1+\kappa^2}{2\kappa}. \tag{5.9}$$

Following again the procedure in section 2.4 we get $\mathcal{N}$, $\mathcal{M}$ in the form (2.11) with

$$S=r\frac{1-\alpha^2\kappa^2}{2\alpha\kappa},\qquad \Delta=r\frac{-\alpha^2+\kappa^2}{2\alpha\kappa},\qquad \sigma=p,$$

$$T=r\frac{1-\beta^2\kappa^2}{2\beta\kappa},\qquad \Omega=r\frac{-\beta^2+\kappa^2}{2\beta\kappa},\qquad \tau=q,$$

$$U_{n,m}=A\alpha^n\beta^m-B\alpha^{-n}\beta^{-m}.$$

Then defining ρ by

$$\rho_{n,m}=\left(\frac{S}{\Delta}\right)^n\left(\frac{T}{\Omega}\right)^m\rho_{0,0}=\left(\frac{\alpha^2\kappa^2-1}{\alpha^2-\kappa^2}\right)^n\left(\frac{\beta^2\kappa^2-1}{\beta^2-\kappa^2}\right)^m\rho_{0,0}, \tag{5.10}$$

we find

$$v_{n,m}=\frac{\frac{U_{n,m}}{U_{0,0}}v_{0,0}\rho_{n,m}/\rho_{0,0}}{1-\frac{\kappa}{1-\kappa^2}\cdot\frac{v_{0,0}}{U_{0,0}}+\frac{\kappa}{1-\kappa^2}\cdot\frac{v_{0,0}}{U_{0,0}}\rho_{n,m}/\rho_{0,0}}=\frac{\frac{1-\kappa^2}{\kappa}U_{n,m}\rho_{n,m}}{1+\rho_{n,m}},$$

and finally

$$u^{1SS}_{n,m}=\frac{A\alpha^n\beta^m(1+\kappa^{-2}\rho_{n,m})+B\alpha^{-n}\beta^{-m}(1+\kappa^2\rho_{n,m})}{1+\rho_{n,m}}. \tag{5.11}$$

5.3. Bilinear form and Casoratian solutions

5.3.1. N-soliton solution. Noting that $\rho_{n,m}$ given by (5.10) is in a 'twisted' form in comparison with (3.14) and (4.14), we first introduce the Möbius transformations for the parameters

$$\alpha^2=-\frac{a-c}{a+c},\qquad \beta^2=-\frac{b-c}{b+c},\qquad \kappa^2=-\frac{k-c}{k+c},$$
$$\Rightarrow\quad p^2=\frac{r^2c^2}{c^2-a^2},\qquad q^2=\frac{r^2c^2}{c^2-b^2}, \tag{5.12}$$

which also contain a new auxiliary parameter c. This brings $\rho_{n,m}$ of (5.10) into the canonical form

$$\rho_{n,m}=\left(\frac{a+k}{a-k}\right)^n\left(\frac{b+k}{b-k}\right)^m\rho_{0,0}. \tag{5.13}$$

In terms of the new form of $\rho_{n,m}$ we can write the 1SS (5.11) as

$$u_{n,m}^{1\text{SS}} = A\alpha^n\beta^m \frac{\psi(n,m,l+1)}{\psi(n,m,l)} + B\alpha^{-n}\beta^{-m}\frac{\psi(n,m,l-1)}{\psi(n,m,l)}, \qquad AB = -\frac{1}{4}r\delta, \tag{5.14}$$

where ψ is as in (2.20).

On the basis of the above 1SS, we propose that the NSS of H3 can be given by

$$u_{n,m}^{\text{NSS}} = A\alpha^n\beta^m \frac{\overline{f}}{f} + B\alpha^{-n}\beta^{-m}\frac{\underline{f}}{f}, \qquad AB = -\frac{1}{4}r\delta, \tag{5.15}$$

where $f = |\widehat{N-1}|_{[\nu]}$ with entries (2.20). We may consider (5.15) as a dependent variable transformation for (5.1).

The solution (5.15) is the same as (I-5.21) with $B = C = 0$. In fact, let $\frac{1}{r}$ in (5.15) equal a, which is the direction parameter for the bar-shift in part I. Then comparing (5.12) with (I-5.1c), using (I-2.32, 5.19) and substituting ρ_i in (I-2.2) by

$$\rho_i = \left(\frac{p+k_i}{p-k_i}\right)^n \left(\frac{q+k_i}{q-k_i}\right)^m \left(\frac{a+k_i}{a-k_i}\right)^l \rho_i^0, \tag{5.16}$$

one finds

$$\vartheta = \alpha^{-n}\beta^{-m}, \qquad V(a)\Big/\prod_j(a-k_j) = \frac{\underline{f}}{f}, \qquad V(-a)\times\prod_j(a-k_j) = \frac{\overline{f}}{f},$$

with the same parameter identification as in (3.20).

5.3.2. Bilinearization-I. After introducing the two bilinear equations

$$\mathcal{B}_1 \equiv 2cf\widetilde{f} + (a-c)\widetilde{\overline{f}}\underline{f} - (a+c)\overline{f}\widetilde{\underline{f}} = 0, \tag{5.17a}$$

$$\mathcal{B}_2 \equiv 2cf\widehat{f} + (b-c)\widehat{\overline{f}}\underline{f} - (b+c)\overline{f}\widehat{\underline{f}} = 0, \tag{5.17b}$$

we can represent H3 (5.1) as

$$\text{H3} \equiv \frac{-\alpha^{4n+2}\beta^{4m+2}(a+c)(b+c)\delta^2 P_1 + 4\alpha^{2n}\beta^{2m}\delta B^2 P_2 + 16(a+c)(b+c)B^4 P_3}{32\alpha^{2n+2}\beta^{2m+2}(a+c)^2(b+c)^2 B^2 f\widetilde{f}\widehat{f}\widehat{\widetilde{f}}},$$

where

$$P_1 = \widehat{\widetilde{\overline{f}}}[(b-c)\widehat{\overline{f}}\mathcal{B}_1 - (a-c)\widetilde{\overline{f}}\mathcal{B}_2] - \overline{f}[(b+c)\widetilde{\overline{f}}\widehat{\mathcal{B}_1} - (a+c)\widehat{\overline{f}}\widetilde{\mathcal{B}_2}],$$
$$P_2 = 2c[(b+c)(b-c)(\widehat{f}\widehat{\widetilde{f}}\mathcal{B}_1 + f\widetilde{f}\widehat{\mathcal{B}_1}) - (a+c)(a-c)(\widetilde{f}\widehat{\widetilde{f}}\mathcal{B}_2 + f\widehat{f}\widetilde{\mathcal{B}_2})],$$
$$P_3 = \widehat{\widetilde{\underline{f}}}[(b+c)\widehat{\underline{f}}\mathcal{B}_1 - (a+c)\widetilde{\underline{f}}\mathcal{B}_2] - \underline{f}[(b-c)\widetilde{\underline{f}}\widehat{\mathcal{B}_1} - (a-c)\widehat{\underline{f}}\widetilde{\mathcal{B}_2}].$$

Thus (5.17) can be considered as a bilinearization of H3, and the final step in constructing the NSS is

Proposition 4. *The Casoratian*

$$f = |\widehat{N-1}|_{[\nu]}, \qquad (\nu = 1, 2 \textit{ or } 3), \tag{5.18}$$

with entries given by ψ *of (2.20), solves the bilinear H3 (5.17).*

Proof. We prove (5.17*a*) in its down-tilde-shifted version

$$2c\underset{\sim}{f}f + (a-c)\overline{f}\underset{\sim}{\underline{f}} - (a+c)\overline{\underset{\sim}{f}}\underline{f} = 0. \tag{5.19}$$

For this equation we use Casoratians w.r.t. hat shift, i.e., $\kappa \equiv 2$ in (A.6). $\utilde{f}$, f, $\overline{f}$, $\utilde{\underline{f}}$, $\utilde{\overline{f}}$ and $\underline{f}$ are given by the formulae (A.6a) with $\mu = 1$, (A.6l) with $\mu = \nu = 3$, (A.6c) with $\mu = 3$, (A.6k) with $\mu = 1$ and $\nu = 3$, (A.6l) with $\mu = 3$ and $\nu = 1$, and (A.6a) with $\mu = 3$, respectively. Then we have

$$
\begin{aligned}
&\frac{1}{|\Gamma_3|}(a-b)^{N-2}(c+b)^{N-2}(c-b)^{N-2}[2c\utilde{f}f + (a-c)\overline{f}\utilde{\underline{f}} - (a+c)\utilde{\overline{f}}\underline{f}] \\
&\quad = -|\widehat{N-2}, \utilde{\psi}(N-2)|_{[2]}|\widehat{N-3}, \underline{\psi}(N-2), \mathring{E}^3\psi(N-2)|_{[2]} \\
&\qquad + |\widehat{N-2}, \mathring{E}^3\psi(N-2)|_{[2]}|\widehat{N-3}, \underline{\psi}(N-2), \utilde{\psi}(N-2)|_{[2]} \\
&\qquad + |\widehat{N-2}, \underline{\psi}(N-2)|_{[2]}|\widehat{N-3}, \utilde{\psi}(N-2), \mathring{E}^3\psi(N-2)|_{[2]} \\
&\quad = 0,
\end{aligned}
$$

where we have made use of lemma 2 in which $\mathbf{B} = (\widehat{N-3})$, and $\mathbf{a} = \psi(N-2)$, $\mathbf{b} = \utilde{\psi}(N-2)$, $\mathbf{c} = \underline{\psi}(N-2)$ and $\mathbf{d} = \mathring{E}^3\psi(N-2)$.

The other bilinear equation (5.17b) can be proved in its down-hat-shifted version in a similar way by taking $\kappa \equiv 1$. □

5.3.3. Bilinearization-II. In fact there is another bilinearization of H3 using (5.15). Consider the bilinear system

$$\mathcal{B}'_1 \equiv (b+c)\widehat{\widetilde{\overline{f}}}\,\overline{f} + (a-c)f\,\widehat{\widetilde{\overline{f}}} - (a+b)\widetilde{\overline{f}}\,\widehat{\overline{f}} = 0, \tag{5.20a}$$

$$\mathcal{B}'_2 \equiv (c-b)\widehat{\widetilde{f}}\,\underline{f} - (a+c)f\,\widehat{\widetilde{\underline{f}}} + (a+b)\widetilde{f}\,\widehat{\underline{f}} = 0, \tag{5.20b}$$

$$\mathcal{B}'_3 \equiv (c-a)(b+c)\widetilde{\overline{f}}\,\widehat{\underline{f}} + (a+c)(b-c)\widehat{\overline{f}}\,\widetilde{\underline{f}} + 2c(a-b)f\,\widehat{\widetilde{f}} = 0. \tag{5.20c}$$

This system is related to H3 through

$$
\begin{aligned}
\mathrm{H3} \equiv \frac{c}{f\widetilde{f}\widehat{f}\widehat{\widetilde{f}}}\Bigg[&A^2\alpha^{2n}\beta^{2m}\frac{\widehat{f}\widehat{\widetilde{f}}\mathcal{B}'_1 - \widetilde{f}\widehat{\widetilde{f}}\mathcal{B}'_2}{(a+c)(b+c)} + B^2\alpha^{-2n}\beta^{-2m}\frac{\underline{\widehat{f}}\widetilde{f}\underline{\mathcal{B}}'_1 - \underline{\widetilde{f}}\widehat{f}\underline{\mathcal{B}}'_2}{(a-c)(b-c)} \\
&+ AB\left(\frac{\widetilde{f}\underline{\widehat{f}}\mathcal{B}'_2 + \widehat{f}\widehat{\widetilde{f}}\underline{\mathcal{B}}'_2}{(a+c)(b-c)} - \frac{\widehat{f}\underline{\widetilde{f}}\mathcal{B}'_1 + \widetilde{f}\widehat{\widetilde{f}}\underline{\mathcal{B}}'_1}{(a-c)(b+c)} - \frac{2(a+b)\widetilde{f}\widehat{f}\mathcal{B}'_3}{(a^2-c^2)(b^2-c^2)}\right)\Bigg].
\end{aligned}
$$

The bilinear system (5.20) shares the same Casoratian solutions (5.17):

Proposition 5. *The Casoratian*

$$f = |\widehat{N-1}|_{[\nu]}, \qquad (\nu = 1, 2 \text{ or } 3), \tag{5.21}$$

with entries given by ψ of (2.20), solves the bilinear H3 (5.20).

Proof. We only prove (5.20a) and (5.20c). (5.20b) is similar to (5.20a). We prove (5.20a) in its down-tilde-shifted version, i.e.,

$$(b+c)\widehat{f}\,\utilde{\overline{f}} + (a-c)\utilde{f}\,\widehat{\overline{f}} - (a+b)f\,\utilde{\widehat{\overline{f}}} = 0. \tag{5.22}$$

We need to use Casoratians w.r.t. bar shift. So we now fix $\kappa \equiv 3$ in (A.6). For $\widehat{f}$, $\utilde{\overline{f}}$, $\utilde{f}$, $\widehat{\overline{f}}$ and $\utilde{\widehat{\overline{f}}}$, we use the formulae (A.6d) with $\mu = 2$, (A.6g) with $\mu = 1$, (A.6b) with $\mu = 1$, (A.6i)

with $\mu = 2$ and (A.6n) with $\mu = 2$ and $\nu = 1$, respectively, and $f = |\widehat{N-1}|_{[3]}$. Then we have

$$\begin{aligned}
&\frac{1}{|\Gamma_2|}(a-c)^{N-2}(b+c)^{N-2}[(b+c)\widehat{\widetilde{f}}\,\underset{\sim}{f} + (a-c)\underset{\sim}{f}\,\widehat{\widehat{f}} - (a+b)f\,\underset{\sim}{\widehat{\widehat{f}}}]\\
&\qquad = -|\widehat{N-2}, \mathring{E}^2\psi(N-1)|_{[3]}|\widetilde{N-1}, \underset{\sim}{\psi}(N-1)|_{[3]}\\
&\qquad\quad + |\widehat{N-2}, \underset{\sim}{\psi}(N-1)|_{[3]}|\widetilde{N-1}, \mathring{E}^2\psi(N-1)|_{[3]}\\
&\qquad\quad - |\widehat{N-1}|_{[3]}|\widetilde{N-1}, \underset{\sim}{\psi}(N-1), \mathring{E}^2\psi(N-1)|_{[3]}\\
&\qquad = 0,
\end{aligned}$$

where we have made use of lemma 2 in which $\mathbf{B} = (\widetilde{N-2})$, and $\mathbf{a} = \psi(0)$, $\mathbf{b} = \psi(N-1)$, $\mathbf{c} = \underset{\sim}{\psi}(N-1)$ and $\mathbf{d} = \mathring{E}^2\psi(N-1)$.

For (5.20c), after a down-hat shift we obtain

$$2c(a-b)\underset{\wedge}{f}\,\widetilde{f} - (a-c)(b+c)\underset{\wedge}{\widetilde{\overline{f}}}\,\underline{f} + (b-c)(a+c)\overline{f}\,\underset{\vartriangle}{\widetilde{f}} = 0. \tag{5.23}$$

In this case we fix $\kappa \equiv 1$ in (A.6). For $\underset{\wedge}{f}$, $\widetilde{f}$, $\underset{\wedge}{\widetilde{\overline{f}}}$, $\underline{f}$, $\overline{f}$ and $\underset{\vartriangle}{\widetilde{f}}$, we use formulae (A.6$b$) with $\mu = 2$, (A.6n) with $\mu = \nu = 3$, (A.6n) with $\mu = 3$ and $\nu = 2$, (A.6b) with $\mu = 3$, (A.6d) with $\mu = 3$, and (A.6m) with $\mu = 2$ and $\nu = 3$, respectively. Then

$$\begin{aligned}
&\frac{1}{|\Gamma_3|}(b-a)^{N-2}(c+a)^{N-2}(c-a)^{N-2}[2c(a-b)\underset{\wedge}{f}\,\widetilde{f} - (a-c)(b+c)\underset{\wedge}{\widetilde{\overline{f}}}\,\underline{f} + (b-c)(a+c)\overline{f}\,\underset{\vartriangle}{\widetilde{f}}]\\
&\qquad = -|\widehat{N-2}, \underset{\wedge}{\psi}(N-1)|_{[1]}|\widetilde{N-2}, \underline{\psi}(N-1), \mathring{E}^3\psi(N-1)|_{[1]}\\
&\qquad\quad + |\widehat{N-2}, \underline{\psi}(N-1)|_{[1]}|\widetilde{N-2}, \underset{\wedge}{\psi}(N-1), \mathring{E}^3\psi(N-1)|_{[1]}\\
&\qquad\quad + |\widehat{N-2}, \mathring{E}^3\psi(N-1)|_{[1]}|\widetilde{N-2}, \underline{\psi}(N-1), \underset{\wedge}{\psi}(N-1)|_{[1]}\\
&\qquad = 0,
\end{aligned}$$

where in lemma 2 we take this time $\mathbf{B} = (\widetilde{N-2})$, and $\mathbf{a} = \psi(0)$, $\mathbf{b} = \underline{\psi}(N-1)$, $\mathbf{c} = \underset{\wedge}{\psi}(N-1)$ and $\mathbf{d} = \mathring{E}^3\psi(N-1)$. □

6. Q1 with a linear background

6.1. *Background solution with $T(x) = x + c$*

Q1 is

$$\mathrm{Q1} \equiv p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}}) - q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) - \delta^2 pq(q-p) = 0. \tag{6.1}$$

With the fixed point defined by $T(x) = x + c$ the side equations for the background are

$$r(u-\widetilde{u})^2 = p(c^2 + \delta^2 r(p-r)), \qquad r(u-\widehat{u})^2 = q(c^2 + \delta^2 r(q-r)). \tag{6.2}$$

After the reparameterization $(p, q) \to (a, b)$ with

$$p = \frac{c^2/r - \delta^2 r}{a^2 - \delta^2}, \qquad q = \frac{c^2/r - \delta^2 r}{b^2 - \delta^2}, \qquad \alpha := pa, \qquad \beta := qb, \tag{6.3}$$

equations (6.2) factorize as in (3.3), and thus the 0SS will be as in (3.5), where, however, we should replace a and b with α and β, respectively.

6.2. 1-soliton solution

The BT for constructing the 1SS is

$$p(u-\overline{u})(\widetilde{u}-\widetilde{\overline{u}})-\varkappa(u-\widetilde{u})(\overline{u}-\widetilde{\overline{u}})=\delta^2 p\varkappa(\varkappa-p), \tag{6.4a}$$

$$\varkappa(u-\widehat{u})(\overline{u}-\widehat{\overline{u}})-q(u-\overline{u})(\widehat{u}-\widehat{\overline{u}})=\delta^2\varkappa q(q-\varkappa). \tag{6.4b}$$

Following the usual procedure we take $u=\alpha n+\beta m+\gamma$ as the 0SS and

$$\overline{u}=\alpha n+\beta m+\gamma+\kappa+v, \tag{6.5}$$

as the 1SS, where v is to be determined. If we now choose

$$\varkappa=\frac{c^2/r-\delta^2 r}{k^2-\delta^2},\qquad \kappa=k\varkappa, \tag{6.6}$$

then we get (2.10) with

$$E=-\varkappa(a+k),\qquad F=-\varkappa(a-k),\qquad G=-\varkappa(b+k),\qquad H=-\varkappa(b-k).$$

Thus if we define

$$\rho_{nm}=\left(\frac{a+k}{a-k}\right)^n\left(\frac{b+k}{b-k}\right)^n\rho_{00}$$

we find the 1SS in the form

$$u=\alpha n+\beta m+\gamma+\kappa\frac{1-\rho_{nm}}{1+\rho_{nm}},$$

where $p,q,\varkappa$ depend on a,b,k as given in (6.3) and (6.6). This is similar to (3.16) except for the more complicated dependence on the parameters a,b,k.

6.3. NSS

After studying the 2SS using Hirota's perturbative method we propose that the NSS is obtained using the following:

$$u_{n,m}^{\mathrm{NSS}}=\alpha n+\beta m+\gamma-(c^2/r-\delta^2 r)\frac{\mathfrak{g}}{f}, \tag{6.7}$$

where

$$f=|\widehat{N-1}|_{[3]},\qquad \mathfrak{g}=|-1,\widetilde{N-1}|_{[3]}, \tag{6.8}$$

with ψ defined by

$$\psi_i(n,m,l)=\varrho_i^{+}(a+k_i)^n(b+k_i)^m(\delta+k_i)^l+\varrho_i^{-}(a-k_i)^n(b-k_i)^m(\delta-k_i)^l, \tag{6.9}$$

The solution (6.7) is consistent with (I-5.11): taking $c=0$ in (6.7) and $A=D=0$, $B=\delta/2$ in (I-5.11), then using (I-2.34c, 5.1b, 5.4) one finds that

$$a=\delta,\qquad S(-a,a)=\sum_j\frac{1}{k_j-a}+\frac{\mathfrak{g}}{f}.$$

However, the additional parameter c (6.7) implies more freedom, cf (6.3) and (I-5.1b).

The functions $f,\mathfrak{g}$ satisfy the following bilinear equations (see proposition below) among others

$$\mathcal{Q}_1\equiv\widehat{\widetilde{\overline{f}}}f(b-\delta)+\widehat{\widetilde{f}\,\overline{f}}(a+\delta)-\widetilde{\overline{f}}\widehat{f}(a+b)=0, \tag{6.10a}$$

$$\mathcal{Q}_2\equiv\widehat{\widetilde{\overline{f}}}f(a-b)+\widetilde{\overline{f}}\widehat{f}(b+\delta)-\widetilde{f}\,\widehat{\overline{f}}(a+\delta)=0, \tag{6.10b}$$

$$\mathcal{Q}_3 \equiv -\widetilde{\overline{f}}\,\widehat{f} + \widetilde{\overline{f}}\,\widehat{\mathfrak{g}}(-a+\delta) + \widetilde{f}\,\widehat{\overline{f}} + \widehat{\overline{f}}\,\widetilde{\mathfrak{g}}(b-\delta) + \overline{f}\,\widehat{\widetilde{\mathfrak{g}}}(a-b) = 0, \tag{6.10c}$$

$$\mathcal{Q}_4 = \widehat{\widetilde{\overline{f}}}\,\mathfrak{g}(a-b) + \widetilde{\overline{f}}\,\widehat{\mathfrak{g}}(a+b) - \widehat{\overline{f}}\,\widetilde{\mathfrak{g}}(a+b) + \overline{f}\,\widehat{\widetilde{\mathfrak{g}}}(-a+b) = 0. \tag{6.10d}$$

We note that $\mathcal{Q}_4$ can also be replaced by

$$\mathcal{Q}_4' = \mathcal{Q}_3 + \mathcal{Q}_4 = \widehat{\overline{f}}\,\widetilde{f} - \widehat{\overline{f}}\,\widetilde{\mathfrak{g}}(a+\delta) - \widetilde{\overline{f}}\,\widehat{f} + \widetilde{\overline{f}}\,\widehat{\mathfrak{g}}(\delta+b) + \widehat{\widetilde{\overline{f}}}\,\mathfrak{g}(a-b) = 0, \tag{6.11}$$

which is similar to $\mathcal{Q}_3$. When the dependent variable transformation (6.7) is substituted into Q1 we find that the result can be expressed in terms of the $\mathcal{Q}_i$ defined above:

$$\text{Q1} = \frac{(c^2/r - \delta^2 r)^3}{(a^2-\delta^2)(b^2-\delta^2)(a-b)(a+\delta)\overline{f}\,f\,\widetilde{f}\,\widehat{f}\,\widehat{\widetilde{f}}} \quad \sum_{i=1}^{4} \mathcal{Q}_i P_i, \tag{6.12}$$

with

$$\begin{aligned} P_1 = (a-b)[&\widetilde{f}\,\widehat{f}\,\mathfrak{g}(-a+b) + f(\widehat{f}\,\widetilde{\mathfrak{g}} - \widetilde{f}\,\widehat{\mathfrak{g}})(a+b) \\ &+ \widetilde{f}\,\widehat{\mathfrak{g}}\,\mathfrak{g}(-a^2+\delta^2) + \widehat{f}\,\widetilde{\mathfrak{g}}\,\mathfrak{g}(b^2-\delta^2) + f\,\widetilde{\mathfrak{g}}\,\widehat{\mathfrak{g}}(a^2-b^2)], \end{aligned} \tag{6.13a}$$

$$P_2 = (a+b)[\widetilde{f}\,\widehat{f}\,\mathfrak{g}(a-b) + f(\widetilde{f}\,\widehat{\mathfrak{g}} - \widehat{f}\,\widetilde{\mathfrak{g}})(b-\delta) + \widetilde{\mathfrak{g}}(\widehat{f}\,\mathfrak{g} - f\,\widehat{\mathfrak{g}})(a-b)(b-\delta)], \tag{6.13b}$$

$$P_3 = (a+b)(a+\delta)[\widetilde{f}\,\widehat{f}\,\mathfrak{g}(a-b) + \widetilde{f}\,f\,\widehat{\mathfrak{g}}(b-\delta) + \widehat{f}\,f\,\widetilde{\mathfrak{g}}(-a+\delta)], \tag{6.13c}$$

$$P_4 = f(a+\delta)[\widetilde{f}\,\widehat{f}(-a+b) + (\widetilde{f}\,\widehat{\mathfrak{g}} - \widehat{f}\,\widetilde{\mathfrak{g}})(a-\delta)(b-\delta)]. \tag{6.13d}$$

Then it remains to prove the following:

Proposition 6. *The Casoratian type determinants $f, \mathfrak{g}$, given in (6.8) with entries given by (6.9) solve the set of bilinear equations (6.10).*

Lemma 3. *By means of lemma 2, the following formulae are zero* ($\mu = 1, 2$),

$$Y_\mu = f|E_\mu\psi(-1), -1, \widetilde{N-2}|_{[3]} + \underline{f}|E_\mu\psi(-1), \widetilde{N-1}|_{[3]} - \mathfrak{g}|E_\mu\psi(-1), \widehat{N-2}|_{[3]}, \tag{6.14a}$$

$$\begin{aligned} Y_3 = {} & |\underset{\sim}{\psi}(-1), -1, \widetilde{N-2}|_{[3]}|\underset{\wedge}{\psi}(-1), \widetilde{N-1}|_{[3]} - |\underset{\wedge}{\psi}(-1), -1, \widetilde{N-2}|_{[3]}|\underset{\sim}{\psi}(-1), \widetilde{N-1}|_{[3]} \\ & + \mathfrak{g}|\underset{\wedge}{\psi}(-1), \underset{\sim}{\psi}(-1), \widetilde{N-2}|_{[3]}; \end{aligned} \tag{6.14b}$$

$$Z_\mu = f|\mathring{E}^\mu\psi(-1), -1, \widetilde{N-2}|_{[3]} + \underline{f}|\mathring{E}^\mu\psi(-1), \widetilde{N-1}|_{[3]} - \mathfrak{g}|\mathring{E}^\mu\psi(-1), \widehat{N-2}|_{[3]}, \tag{6.15a}$$

$$\begin{aligned} Z_3 = {} & |\mathring{E}^1\psi(-1), -1, \widetilde{N-2}|_{[3]}|\mathring{E}^2\psi(-1), \widetilde{N-1}|_{[3]} \\ & - |\mathring{E}^2\psi(-1), -1, \widetilde{N-2}|_{[3]}|\mathring{E}^1\psi(-1), \widetilde{N-1}|_{[3]} \\ & + \mathfrak{g}|\mathring{E}^2\psi(-1), \mathring{E}^1\psi(-1), \widetilde{N-2}|_{[3]}. \end{aligned} \tag{6.15b}$$

Proof. For the above formulae, we can use lemma 2 by respectively taking

$\mathbf{B} = (\widetilde{N-2})$, $\quad(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = (E_\mu\psi(-1), \psi(-1), \psi(0), \psi(N-1))$,

$\mathbf{B} = (\widetilde{N-2})$, $\quad(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = (\underset{\wedge}{\psi}(-1), \underset{\sim}{\psi}(-1), \psi(-1), \psi(N-1))$;

$\mathbf{B} = (\widetilde{N-2})$, $\quad(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = (\mathring{E}^\mu\psi(-1), \psi(-1), \psi(0), \psi(N-1))$,

$\mathbf{B} = (\widetilde{N-2})$, $\quad(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = (\mathring{E}^2\psi(-1), \mathring{E}^1\psi(-1), \psi(-1), \psi(N-1))$. □

Proof.

Proof for (6.10a). The down-bar shifted (6.10a) is nothing but $\mathcal{B}_2'$ of H3 with $c = \delta$.

Proof for (6.10b). By a down-tilde shift (6.10*b*) is written as

$$(a-b)\underset{\sim}{f}\,\widehat{\overline{f}} + (\delta+b)\overline{f}\,\widehat{\underset{\sim}{f}} - (a+\delta)\widehat{\overline{\underset{\sim}{f}}}\,f = 0. \tag{6.16}$$

We fix $\kappa \equiv 2$ and $c = \delta$ in (A.6). For $\underset{\sim}{f}, \widehat{\overline{f}}, \overline{f}, \widehat{\underset{\sim}{f}}$ and $\widehat{\overline{\underset{\sim}{f}}}$, we use the formulae (A.6*b*) with $\mu = 1$, (A.6*i*) with $\mu = 3$, (A.6*d*) with $\mu = 3$, (A.6*g*) with $\mu = 3$ and (A.6*n*) with $\mu = 3$ and $\nu = 1$, respectively, and $f = |\widehat{N-1}|_{[2]}$. Then we have

$$\begin{aligned}
&\frac{1}{|\Gamma_3|}(a-b)^{N-2}(\delta+b)^{N-2}\big[(a-b)\underset{\sim}{f}\,\widehat{\overline{f}} + (\delta+b)\overline{f}\,\widehat{\underset{\sim}{f}} - (a+\delta)\widehat{\overline{\underset{\sim}{f}}}\,f\big]\\
&\quad = |\widehat{N-2}, \underset{\sim}{\psi}(N-1)|_{[2]}|\widetilde{N-1}, \mathring{E}^3\psi(N-1)|_{[2]}\\
&\qquad - |\widehat{N-2}, \mathring{E}^3\psi(N-1)|_{[2]}|\widetilde{N-1}, \underset{\sim}{\psi}(N-1)|_{[2]}\\
&\qquad - |\widehat{N-1}|_{[2]}|\widetilde{N-1}, \underset{\sim}{\psi}(N-1), \mathring{E}^3\psi(N-1)|_{[2]}\\
&\quad = 0,
\end{aligned}$$

where we have made use of lemma 2 with

$$\mathbf{B} = (\widetilde{N-2}), \qquad (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = (\psi(0), \psi(N-1), \underset{\sim}{\psi}(N-1), \mathring{E}^3\psi(N-1)).$$

Proof for (6.10c). By a down-tilde-hat shift (6.10*c*) is written as

$$-\overline{\underset{\wedge}{f}}[\underset{\sim}{f} + (a-\delta)\underset{\sim}{\mathfrak{g}}] + \overline{\underset{\sim}{f}}[\underset{\wedge}{f} + (b-\delta)\underset{\wedge}{\mathfrak{g}}] + (a-b)\overline{\underset{\hat{\sim}}{f}}\,\mathfrak{g} = 0. \tag{6.17}$$

This time we fix $\kappa \equiv 3$ and $c = \delta$ in (A.27). For $\overline{\underset{\wedge}{f}}, \underset{\sim}{f}, \underset{\sim}{\mathfrak{g}}, \overline{\underset{\sim}{f}}, \underset{\wedge}{f}, \underset{\wedge}{\mathfrak{g}}$ and $\overline{\underset{\hat{\sim}}{f}}$, we use the formulae (A.27*b*) with $\mu = 2$, (A.27*a*) with $\mu = 1$, (A.27*c*) with $\mu = 1$, (A.27$\hat{b}$) with $\mu = 1$, (A.27*a*) with $\mu = 2$, (A.27*c*) with $\mu = 2$, and (A.27*g*), respectively. Then we have

$$\begin{aligned}
&-\overline{\underset{\wedge}{f}}[\underset{\sim}{f} + (a-\delta)\underset{\sim}{\mathfrak{g}}] + \overline{\underset{\sim}{f}}[\underset{\wedge}{f} + (b-\delta)\underset{\wedge}{\mathfrak{g}}] + (a-b)\overline{\underset{\hat{\sim}}{f}}\,\mathfrak{g}\\
&\quad = (a-\delta)^2Y_1 - (b-\delta)^2Y_2 + (a-\delta)^2(b-\delta)^2Y_3
\end{aligned}$$

with Y_j defined in (6.14), which are zero in the light of lemma 3.

Proof for (6.10d). Using (6.10*c*) to eliminate the term $(a-b)\overline{f}\widehat{\widetilde{\mathfrak{g}}}$ from (6.10*d*), we obtain

$$\widehat{\overline{f}}[\widetilde{f} - (a+\delta)\widetilde{\mathfrak{g}}] - \widetilde{\overline{f}}[\widehat{f} - (b+\delta)\widehat{\mathfrak{g}}] + (a-b)\widehat{\widetilde{\overline{f}}}\,\mathfrak{g} = 0. \tag{6.18}$$

We fix $\kappa \equiv 3$ and $c = \delta$ in (A.27), and use formulae (A.27*d*), (A.27*e*), (A.27*f*) and (A.27*h*). Then it turns out that

$$\begin{aligned}
&\widehat{\overline{f}}[\widetilde{f} - (a+\delta)\widetilde{\mathfrak{g}}] - \widetilde{\overline{f}}[\widehat{f} - (b+\delta)\widehat{\mathfrak{g}}] + (a-b)\widehat{\widetilde{\overline{f}}}\,\mathfrak{g}\\
&\quad = (a+\delta)^2Z_1 - (b+\delta)^2Z_2 - (a+\delta)^2(b+\delta)^2Z_3
\end{aligned}$$

with Z_j defined in (6.15), which are zero in the light of lemma 3.

Thus we have completed the proof for all bilinear equations. □

7. Q1 with power background

7.1. Background solution with $T(x) = -x + c$

With fixed point defined by $T(x) = -x + c$ we use the reparameterization

$$p = \tfrac{1}{2}r(1 - \cosh(\alpha')) = -\tfrac{1}{4}r(1-\alpha)^2/\alpha, \tag{7.1}$$

$$q = \tfrac{1}{2} r(1-\cosh(\beta')) = -\tfrac{1}{4} r(1-\beta)^2/\beta, \tag{7.2}$$

$$x_{nm} = A\,\mathrm{e}^{y_{nm}} + B\,\mathrm{e}^{y_{nm}} + c/2, \qquad \text{where} \quad AB = \delta^2 r^2/16, \tag{7.3}$$

and this leads to equations that factorize as in (5.4). Thus we get power-type background solutions for u:

$$u = \tfrac{1}{2}c + A\alpha^n\beta^m + B\alpha^{-n}\beta^{-m}, \qquad AB = \delta^2 r^2/16. \tag{7.4}$$

Here c is related to translation freedom and r to scaling freedom; in the following we take $c = 0$.

7.2. 1-soliton solution for Q1

In order to derive the 1SS we use the BT (6.4) with the seed solution (7.4) with

$$\overline{u} = \overline{u}_0 + v, \tag{7.5}$$

where $\overline{u}_0$ is the bar-shifted background solution (7.4):

$$\overline{u}_0 = A\alpha^n\beta^m\kappa + B\alpha^{-n}\beta^{-m}\kappa^{-1}, \qquad AB = \delta^2 r^2/16 \tag{7.6}$$

and κ is defined through

$$\varkappa = -\frac{r}{4}(1-\kappa)^2/\kappa. \tag{7.7}$$

If we now define

$$U_{n,m} = A\alpha^n\beta^m - B\alpha^{-n}\beta^{-m} \tag{7.8}$$

then it is straightforward to derive (2.11) from (6.4) with

$$S = \frac{r(1-\alpha)(1-\kappa)(1-\alpha\kappa)}{4\alpha\kappa}, \qquad \Delta = \frac{r(1-\alpha)(1-\kappa)(\alpha-\kappa)}{4\alpha\kappa}, \qquad \sigma = p, \tag{7.9}$$

$$T = \frac{r(1-\beta)(1-\kappa)(1-\beta\kappa)}{4\beta\kappa}, \qquad \Omega = \frac{r(1-\beta)(1-\kappa)(\beta-\kappa)}{4\beta\kappa}, \qquad \tau = q. \tag{7.10}$$

On the basis of this we define ρ as usual by

$$\rho_{n,m} = \left(\frac{S}{\Delta}\right)^n \left(\frac{T}{\Omega}\right)^m \rho_{0,0} = \left(\frac{1-\alpha\kappa}{\alpha-\kappa}\right)^n \left(\frac{1-\beta\kappa}{\beta-\kappa}\right)^m \rho_{0,0}, \tag{7.11}$$

where $\rho_{0,0}$ is some constant. Then it follows that

$$v_{n,m} = \frac{\frac{1-\kappa^2}{\kappa} U_{n,m}\rho_{n,m}}{1+\rho_{n,m}}, \tag{7.12}$$

and finally we obtain the 1-soliton for Q1:

$$u_{n,m}^{1SS} = \overline{u}_0 + v_{n,m} \tag{7.13}$$

$$= \frac{A\alpha^n\beta^m(\kappa+\kappa^{-1}\rho_{n,m}) + B\alpha^{-n}\beta^{-m}(\kappa^{-1}+\kappa\rho_{n,m})}{1+\rho_{n,m}} \tag{7.14}$$

$$= \frac{A'\alpha^n\beta^m(1+\kappa^{-2}\rho_{n,m}) + B'\alpha^{-n}\beta^{-m}(1+\kappa^2\rho_{n,m})}{1+\rho_{n,m}}, \tag{7.15}$$

where $A'B' = AB = \delta^2 r^2/16$.

7.3. Bilinearization

In order to get ρ of (7.11) into a nicer form we use the Möbius transformations

$$\alpha=\frac{a-c}{a+c},\qquad \beta=\frac{b-c}{b+c},\qquad \kappa=\frac{k-c}{k+c},$$
$$\Rightarrow\quad p=\frac{rc^2}{a^2-c^2},\qquad q=\frac{rc^2}{b^2-c^2},\tag{7.16}$$

which leads to the canonical form

$$\rho_{n,m}=\left(\frac{a+k}{a-k}\right)^n\left(\frac{b+k}{b-k}\right)^m\rho_{0,0}.\tag{7.17}$$

With the above Möbius transformations Q1 (6.1) can be written as

$$\mathrm{Q1}\equiv(b^2-c^2)(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-(a^2-c^2)(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})-\frac{r^2\delta^2c^4(a^2-b^2)}{(a^2-c^2)(b^2-c^2)}=0.\tag{7.18}$$

Using (7.17) and rearranging the parameters A and B we write the 1SS (7.15) as

$$u^{\mathrm{1SS}}_{n,m}=A\alpha^n\beta^m\frac{\psi(n,m,l+2)}{\psi(n,m,l)}+B\alpha^{-n}\beta^{-m}\frac{\psi(n,m,l-2)}{\psi(n,m,l)},\qquad AB=\delta^2r^2/16,\tag{7.19}$$

where ψ is as in (2.20).

This structure motivates us to bilinearize Q1 through the following transformation,

$$u^{\mathrm{NSS}}_{n,m}=A\alpha^n\beta^m\frac{\overline{\overline{f}}}{f}+B\alpha^{-n}\beta^{-m}\frac{\underline{\underline{f}}}{f},\qquad AB=\delta^2r^2/16,\tag{7.20}$$

where $f=|\widehat{N-1}|_{[\nu]}$ ($\nu=1,2,$ or 3) with entries (2.20) and the bar-shift is the shift in the third index l, as discussed in section 2.5.

The solution (7.20) with $r=1$ is the same as (I-5.11) with $B=0$. In fact, by a comparison (7.16) with (I-3.17) one has

$$\rho(a)=\alpha^{-n}\beta^{-m},\qquad \rho(-a)=\alpha^n\beta^m;$$

and substituting ρ_i in (I-2.2) by (5.16) we find from (I-3.17) that

$$(1+2aS(-a,-a))\times\left(\prod(k_j-a)^2\right)=\frac{\overline{\overline{f}}}{f},\qquad (1-2aS(a,a))\Big/\left(\prod(k_j-a)^2\right)=\frac{\underline{\underline{f}}}{f}.$$

Surprisingly Q1 can also be bilinearized with the same bilinear equations as H3, namely (5.17), and we have already shown in proposition 4 that $f=|\widehat{N-1}|_{[\nu]}$ solves the $\mathcal{B}_i$ in (5.17).

The representation of Q1 (7.18) in terms of $\mathcal{B}_i$ is as follows:

$$\mathrm{Q1}\equiv\frac{\alpha^{4n+2}\beta^{4m+2}(a+c)^2(b+c)^2A^2\overline{P}_1-\alpha^{2n}\beta^{2m}\delta^2/4P_2-(a+c)^2(b+c)^2B^2\underline{P}_1}{\alpha^{2n+1}\beta^{2m+1}(a+c)^2(b+c)^2f\widetilde{f}\widehat{f}\widehat{\widetilde{f}}},$$

where

$$P_1=Y\widetilde{Y}-X\widehat{X},\qquad X=\mathcal{B}_1-2cf\widetilde{f},\qquad Y=\mathcal{B}_2-2cf\widehat{f},$$

$$\begin{aligned}P_2=&-(a+c)(a-c)(b+c)^2(\overline{X\underline{\widehat{X}}}-4c^2\overline{f\widetilde{f}\,\underline{\widehat{f}}\,\underline{\widehat{\widetilde{f}}}})\\&-(a+c)(a-c)(b-c)^2(\underline{X}\,\widehat{\overline{\widehat{X}}}-4c^2\underline{f}\,\underline{\widetilde{f}}\,\widehat{\overline{\widehat{f}}}\,\widehat{\overline{\widehat{\widetilde{f}}}})\\&+4c^2(b+c)(b-c)(X\widehat{X}-4c^2f\widetilde{f}\widehat{f}\widehat{\widetilde{f}})\\&+(b+c)(b-c)(a+c)^2(\overline{Y\underline{\widetilde{Y}}}-4c^2\overline{f\widehat{f}\,\underline{\widetilde{f}}\,\underline{\widehat{\widetilde{f}}}})\\&+(b+c)(b-c)(a-c)^2(\underline{Y}\,\widetilde{\overline{\widetilde{Y}}}-4c^2\underline{f}\,\underline{\widehat{f}}\,\widetilde{\overline{\widetilde{f}}}\,\widehat{\widetilde{\overline{\widetilde{f}}}})\\&-4c^2(a+c)(a-c)(Y\widetilde{Y}-4c^2f\widehat{f}\widetilde{f}\widehat{\widetilde{f}}).\end{aligned}$$

8. Conclusions

In this paper, companion to [1] we have analyzed the soliton solutions to the models H1, H2, H3 and Q1 in the ABS list [2] of partial difference equations. Our method is constructive, progressing in each case from background solution to 1SS to NSS and bilinearization.

Our approach is fairly algorithmic, and as such we hope that it will be usable also for other models with multidimensional consistency. One interesting feature is that in some cases we need several bilinear equations, some of which seem to have the same continuum limit. This is a reminder that there are several ways to discretize a derivative.

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Appendix. Formulae for Casoratians

We define Casoratians

$$f_{[\nu]} = |\widehat{N-1}|_{[\nu]}, \qquad g_{[\nu]} = |\widehat{N-2}, N|_{[\nu]}, \qquad h_{[\nu]} = |\widehat{N-2}, N+1|_{[\nu]},$$

where the matrix entries are given by the function (2.20), i.e.,

$$\psi_i(n, m, l) = \varrho_i^+ (c + k_i)^l (a + k_i)^n (b + k_i)^m + \varrho_i^- (c - k_i)^l (a - k_i)^n (b - k_i)^m. \tag{A.1}$$

We introduce notations

$$\Gamma_{\nu,i} = \alpha_\nu^2 - k_i^2, \qquad \Gamma_\nu = \mathrm{Diag}(\Gamma_{\nu,1}, \Gamma_{\nu,2}, \cdots, \Gamma_{\nu,N}) \quad (\nu = 1, 2, 3), \tag{A.2}$$

where

$$\alpha_1 \equiv a, \qquad \alpha_2 \equiv b, \qquad \alpha_3 \equiv c, \tag{A.3}$$

and define the operator $\mathring{E}^\nu$ as

$$\mathring{E}^\nu \psi = \Gamma_\nu^{-1} E^\nu \psi \quad (\nu = 1, 2, 3). \tag{A.4}$$

The column vector ψ satisfies

$$(\alpha_\mu - \alpha_\kappa)\psi = (E^\mu - E^\kappa)\psi, \tag{A.5a}$$

$$(\alpha_\mu + \alpha_\nu)\mathring{E}^\mu E_\nu \psi = (\mathring{E}^\mu + E_\nu)\psi, \tag{A.5b}$$

$$E_\mu \psi = [E_\kappa - (\alpha_\mu - \alpha_\kappa)(E_\kappa)^2 + (\alpha_\mu - \alpha_\kappa)^2 E_\mu (E_\kappa)^2]\psi, \tag{A.5c}$$

$$E_\mu \psi = [E_\kappa - (\alpha_\mu - \alpha_\kappa)(E_\kappa)^2 + (\alpha_\mu - \alpha_\kappa)^2 (E_\kappa)^3 - (\alpha_\mu - \alpha_\kappa)^3 E_\mu (E_\kappa)^3]\psi. \tag{A.5d}$$

For the Casoratian $f_{[\kappa]}$, we have

$$-(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu f_{[\kappa]} = |\widehat{N-2}, E_\mu \psi(N-2)|_{[\kappa]}, \tag{A.6a}$$

$$(\alpha_\mu - \alpha_\kappa)^{N-1} E_\mu f_{[\kappa]} = |\widehat{N-2}, E_\mu \psi(N-1)|_{[\kappa]}, \tag{A.6b}$$

$$(\alpha_\mu + \alpha_\kappa)^{N-2} E^\mu f_{[\kappa]} = |\Gamma_\mu||\widehat{N-2}, \mathring{E}^\mu \psi(N-2)|_{[\kappa]}, \quad (\mu \neq \kappa), \tag{A.6c}$$

$$(\alpha_\mu + \alpha_\kappa)^{N-1} E^\mu f_{[\kappa]} = |\Gamma_\mu||\widehat{N-2}, \mathring{E}^\mu \psi(N-1)|_{[\kappa]}, \quad (\mu \neq \kappa); \tag{A.6d}$$

$$E^{\kappa} f_{[\kappa]} = |\widetilde{N}|_{[\kappa]}, \tag{A.6e}$$

$$E_{\kappa} f_{[\kappa]} = |-1, \widehat{N-2}|_{[\kappa]}, \tag{A.6f}$$

$$-(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu E^{\kappa} f_{[\kappa]} = |\widetilde{N-1}, E_\mu \psi(N-1)|_{[\kappa]}, \tag{A.6g}$$

$$(\alpha_\mu - \alpha_\kappa)^{N-1} E_\mu E_\kappa f_{[\kappa]} = |-1, \widehat{N-3}, E_\mu \psi(N-2)|_{[\kappa]}, \tag{A.6h}$$

$$(\alpha_\mu + \alpha_\kappa)^{N-2} E^{\mu} E^{\kappa} f_{[\kappa]} = |\Gamma_\mu||\widetilde{N-1}, \mathring{E}^{\mu} \psi(N-1)|_{[\kappa]} \qquad (\mu \neq \kappa), \tag{A.6i}$$

$$(\alpha_\mu + \alpha_\kappa)^{N-1} E^{\mu} E_\kappa f_{[\kappa]} = |\Gamma_\mu|| -1, \widehat{N-3}, \mathring{E}^{\mu} \psi(N-2)|_{[\kappa]} \qquad (\mu \neq \kappa); \tag{A.6j}$$

$$(\alpha_\mu - \alpha_\nu)(\alpha_\mu - \alpha_\kappa)^{N-2}(\alpha_\nu - \alpha_\kappa)^{N-2} E_\mu E_\nu f_{[\kappa]} = |\widehat{N-3}, E_\nu \psi(N-2), E_\mu \psi(N-2)|_{[\kappa]}, \tag{A.6k}$$

$$(\alpha_\mu + \alpha_\nu)(\alpha_\mu + \alpha_\kappa)^{N-2}(\alpha_\nu - \alpha_\kappa)^{N-2} E^{\mu} E_\nu f_{[\kappa]} \\ = |\Gamma_\mu||\widehat{N-3}, E_\nu \psi(N-2), \mathring{E}^{\mu} \psi(N-2)|_{[\kappa]} \quad (\mu \neq \kappa); \tag{A.6l}$$

$$(\alpha_\mu - \alpha_\nu)(\alpha_\mu - \alpha_\kappa)^{N-2}(\alpha_\nu - \alpha_\kappa)^{N-2} E_\mu E_\nu E^{\kappa} f_{[\kappa]} \\ = |\widetilde{N-2}, E_\nu \psi(N-1), E_\mu \psi(N-1)|_{[\kappa]}, \tag{A.6m}$$

$$(\alpha_\mu + \alpha_\nu)(\alpha_\mu + \alpha_\kappa)^{N-2}(\alpha_\nu - \alpha_\kappa)^{N-2} E^{\mu} E_\nu E^{\kappa} f_{[\kappa]} \\ = |\Gamma_\mu||\widetilde{N-2}, E_\nu \psi(N-1), \mathring{E}^{\mu} \psi(N-1)|_{[\kappa]} \qquad (\mu \neq \kappa). \tag{A.6n}$$

For $g_{[\kappa]}$ we have

$$-(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu [g_{[\kappa]} + (\alpha_\mu - \alpha_\kappa) f_{[\kappa]}] = |\widehat{N-3}, N-1, E_\mu \psi(N-2)|_{[\kappa]}, \tag{A.7a}$$

$$(\alpha_\mu + \alpha_\kappa)^{N-2} E^{\mu} [g_{[\kappa]} - (\alpha_\mu + \alpha_\kappa) f_{[\kappa]}] = |\Gamma_\mu||\widehat{N-3}, N-1, \mathring{E}^{\mu} \psi(N-2)|_{[\kappa]} \quad (\mu \neq \kappa). \tag{A.7b}$$

For $h_{[\kappa]}$,

$$-(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu [h_{[\kappa]} + (\alpha_\mu - \alpha_\kappa) g_{[\kappa]}] = |\widehat{N-3}, N, E_\mu \psi(N-2)|_{[\kappa]}, \tag{A.8a}$$

$$(\alpha_\mu + \alpha_\kappa)^{N-2} E^{\mu} [h_{[\kappa]} - (\alpha_\mu + \alpha_\kappa) g_{[\kappa]}] = |\Gamma_\mu||\widehat{N-3}, N, \mathring{E}^{\mu} \psi(N-2)|_{[\kappa]} \quad (\mu \neq \kappa). \tag{A.8b}$$

These formulae, (A.5)–(A.8), are valid for $\mu, \nu, \kappa = 1, 2, 3$ except when otherwise indicated.

Now we prove the above formulae. Formulae (A.5) can directly be verified. In (A.5)–(A.8), those only containing down shifts E_μ and E_ν can be proved by using the relations (A.5). Since a, b and c appear in ψ_i equivalently, we take $\mu = 1$, $\nu = 2$ and $\kappa = 3$ in the following proof.

From (A.5a) we have

$$(a-c)\underset{\sim}{\psi}(l) = \psi(l) - \underset{\sim}{\psi}(l+1), \tag{A.9}$$

$$(b-c)\underset{\wedge}{\psi}(l) = \psi(l) - \underset{\wedge}{\psi}(l+1), \tag{A.10}$$

$$(a-b)\underset{\hat{\sim}}{\psi}(l) = \underset{\wedge}{\psi}(l) - \underset{\sim}{\psi}(l). \tag{A.11}$$

Using (A.9) for $f_{[3]}$ we first find

$$(a-c)\underset{\sim}{f}_{[3]} = |(a-c)\underset{\sim}{\psi}(0), \underset{\sim}{\psi}(1), \ldots, \underset{\sim}{\psi}(N-1)|_{[3]} = |\psi(0), \underset{\sim}{\psi}(1), \ldots, \underset{\sim}{\psi}(N-1)|_{[3]}$$

and repeating this $N-1$ times we obtain

$$(a-c)^{N-1}\utilde{f}_{[3]} = |\psi(0),\ldots,\psi(N-2),\utilde{\psi}(N-1)|_{[3]} = |\widehat{N-2},\utilde{\psi}(N-1)|_{[3]}.$$

This is formula (A.6b). Again using (A.9) and expressing $\utilde{\psi}(N-1)$ through $\psi(N-2)$ and $\utilde{\psi}(N-2)$ in the above formula yields

$$-(a-c)^{N-2}f_{[3]} = |\widehat{N-2},\utilde{\psi}(N-2)|_{[3]}, \tag{A.12}$$

which is (A.6a). For $\underset{\hat{}}{\utilde{f}}_{[3]}$ we first used (A.10) from the above formula to obtain

$$-(b-c)^{N-2}(a-c)^{N-2}\underset{\hat{}}{\utilde{f}}_{[3]} = |\widehat{N-3},\underset{\hat{}}{\psi}(N-2),\underset{\hat{}}{\utilde{\psi}}(N-2)|_{[3]}$$

then from (A.11) we obtain

$$(a-b)(a-c)^{N-2}(b-c)^{N-2}\underset{\hat{}}{\utilde{f}} = |\widehat{N-3},\underset{\hat{}}{\psi}(N-2),\utilde{\psi}(N-2)|_{[3]}, \tag{A.13}$$

which is (A.6k).

For $g_{[3]} = |\widehat{N-2},N|_{[3]}$ we find first

$$(a-c)^{N-2}\utilde{g}_{[3]} = |\widehat{N-3},\utilde{\psi}(N-2),\utilde{\psi}(N)|_{[3]}.$$

Using (A.5c) to rewrite the last column we obtain

$$(a-c)^{N-2}\utilde{g}_{[3]} = -|\widehat{N-3},N-1,\utilde{\psi}(N-2)|_{[3]} + (a-c)|\widehat{N-2},\utilde{\psi}(N-2)|_{[3]}, \tag{A.14}$$

which can also be stated as

$$-(a-c)^{N-2}[\utilde{g}_{[3]} + (a-c)\utilde{f}_{[3]}] = |\widehat{N-3},N-1,\utilde{\psi}(N-2)|_{[3]}, \tag{A.15}$$

i.e., (A.7a). Formula (A.8a) can be derived similarly by using (A.5d).

To show how to derive those formulae with up shifts in (A.6)–(A.8), we introduce two auxiliary functions:

$$\textbf{II}:\quad \phi_i(n,m,l) = \varrho_i^+(c+k_i)^l(a+k_i)^n(b-k_i)^{-m} + \varrho_i^-(c-k_i)^l(a-k_i)^n(b+k_i)^{-m}, \tag{A.16a}$$

$$\textbf{III}:\quad \omega_i(n,m,l) = \varrho_i^+(c+k_i)^l(a-k_i)^{-n}(b+k_i)^m + \varrho_i^-(c-k_i)^l(a+k_i)^{-n}(b-k_i)^m. \tag{A.16b}$$

Casoratians f, g and h (w.r.t. bar-shift) with column vectors $\phi = (\phi_1,\ldots,\phi_N)^T$ and $\omega = (\omega_1,\ldots,\omega_N)^T$ are denoted by $f_{\mathrm{II}[3]}, g_{\mathrm{II}[3]}, h_{\mathrm{II}[3]}$ and $f_{\mathrm{III}[3]}, g_{\mathrm{III}[3]}, h_{\mathrm{III}[3]}$, respectively. They are related to $f_{[3]}$, $g_{[3]}$ and $h_{[3]}$ through

$$f_{[3]} = |\Gamma_2|^m f_{\mathrm{II}[3]} = |\Gamma_1|^n f_{\mathrm{III}[3]}, \tag{A.17a}$$

$$g_{[3]} = |\Gamma_2|^m g_{\mathrm{II}[3]} = |\Gamma_1|^n g_{\mathrm{III}[3]}, \tag{A.17b}$$

$$h_{[3]} = |\Gamma_2|^m h_{\mathrm{II}[3]} = |\Gamma_1|^n h_{\mathrm{III}[3]}, \tag{A.17c}$$

which follow from $\psi = \Gamma_2^m\phi = \Gamma_1^n\omega$.

ϕ satisfies

$$(a-c)\utilde{\phi}(l) = \phi(l) - \utilde{\phi}(l+1), \tag{A.18}$$

$$(b+c)\widehat{\phi}(l) = \phi(l) + \widehat{\phi}(l+1), \tag{A.19}$$

$$(a+b)\utilde{\widehat{\phi}}(l) = \widehat{\phi}(l) + \utilde{\phi}(l). \tag{A.20}$$

Similar to the previous case of ψ, using these relations we can obtain

$$(b+c)^{N-2}\widehat{f}_{\mathrm{II}[3]} = |\widehat{N-2}, \widehat{\phi}(N-2)|_{\mathrm{II}[3]}, \tag{A.21a}$$

$$(b+c)^{N-1}\widehat{f}_{\mathrm{II}[3]} = |\widehat{N-2}, \widehat{\phi}(N-1)|_{\mathrm{II}[3]}, \tag{A.21b}$$

$$-(a+b)(a-c)^{N-2}(b+c)^{N-2}\underset{\sim}{\widehat{f}}_{\mathrm{II}[3]} = |\widehat{N-3}, \widehat{\phi}(N-2), \underset{\sim}{\phi}(N-2)|_{\mathrm{II}[3]}; \tag{A.21c}$$

$$(b+c)^{N-2}\widehat{\widehat{f}}_{\mathrm{II}[3]} = |\widetilde{\widehat{N-1}}, \widehat{\phi}(N-1)|_{\mathrm{II}[3]}, \tag{A.21d}$$

$$-(a+b)(a-c)^{N-2}(b+c)^{N-2}\underset{\sim}{\widehat{\widehat{f}}}_{\mathrm{II}[3]} = |\widetilde{\widehat{N-2}}, \widehat{\phi}(N-1), \underset{\sim}{\phi}(N-1)|_{\mathrm{II}[3]}, \tag{A.21e}$$

$$(b+c)^{N-1}\underset{-}{\widehat{f}}_{\mathrm{II}[3]} = |-1, \widehat{N-3}, \widehat{\phi}(N-2)|_{\mathrm{II}[3]}; \tag{A.21f}$$

$$(b+c)^{N-2}[\widehat{g}_{\mathrm{II}[3]} - (b+c)\widehat{f}_{\mathrm{II}[3]}] = |\widehat{N-3}, N-1, \widehat{\phi}(N-2)|_{\mathrm{II}[3]}, \tag{A.21g}$$

$$(b+c)^{N-2}[\widehat{h}_{\mathrm{II}[3]} - (b+c)\widehat{g}_{\mathrm{II}[3]}] = |\widehat{N-3}, N, \widehat{\phi}(N-2)|_{\mathrm{II}[3]}. \tag{A.21h}$$

Now, noting that $f_{[3]} = |\Gamma_2|^m f_{\mathrm{II}[3]}$ and $\psi = \Gamma_2^m\phi$, we have

$$(b+c)^{N-2}\widehat{f}_{[3]} = (b+c)^{N-2}|\Gamma_2|^{m+1}\widehat{f}_{\mathrm{II}[3]} = |\Gamma_2||\psi(0), \ldots, \psi(N-2), \mathring{E}^2\psi(N-2)|_{[3]}, \tag{A.22}$$

where the operator $\mathring{E}^2$ is defined as (A.4), The above formula is just (A.6*c*) for $\mu = 2$. From the rest of (A.21) we can get other formulae with an up-hat shift in (A.6)–(A.8), where for (A.6*l*) and (A.6*n*) ($\mu = 2, \nu = 1$) we need to use (A.5*b*). Besides, if we take a down-hat shift for (A.22) and rewrite the last column by using (A.5*b*) (with $\mu = \nu = 2$), then we get

$$2b(b+c)^{N-2}(b-c)^{N-2}f_{[3]} = |\Gamma_2||\widehat{N-3}, \underset{\wedge}{\psi}(N-2), \mathring{E}^2\psi(N-2)|_{[\kappa]},$$

i.e., formula (A.6*l*) of the case $\mu = \nu = 2$.

Next, using ω we can derive these formulae with an up-tilde shift in (A.6)–(A.8). ω satisfies

$$(a+c)\widetilde{\omega}(l) = \omega(l) + \widetilde{\omega}(l+1), \tag{A.23}$$

$$(b-c)\underset{\wedge}{\omega}(l) = \omega(l) - \underset{\wedge}{\omega}(l+1), \tag{A.24}$$

$$(a+b)\underset{\wedge}{\widetilde{\omega}}(l) = \underset{\wedge}{\omega}(l) + \widetilde{\omega}(l), \tag{A.25}$$

and from these relations we can have formulae for $\widetilde{f}_{\mathrm{III}[3]}, \widetilde{\widetilde{f}}_{\mathrm{III}[3]}$, etc. Then by using operator $\mathring{E}^1$, these formulae will generate those of (A.6)–(A.8) with an up-tilde shift.

Thus we can get all formulae for f, g and h given by (A.6)–(A.8). We note that using the auxiliary Casoratians with

$$\textbf{IV}: \quad \sigma_i(n,m,l) = \varrho_i^+(c+k_i)^l(a-k_i)^{-n}(b-k_i)^{-m} + \varrho_i^-(c-k_i)^l(a+k_i)^{-n}(b+k_i)^{-m}, \tag{A.26}$$

one can get some formulae with double up-shifts for f, g and h.

Next we list some alternative formulae for $f = |\widehat{N-1}|_{[3]}$ and also formulae for $\mathfrak{g} = |-1, \widetilde{N-1}|_{[3]}$ (with $\mu = 1, 2$):

$$E_\mu f = E_3 f - (\alpha_\mu - \alpha_3)|E_\mu\psi(-1), \widehat{N-2}|_{[3]}, \tag{A.27a}$$

$$E_\mu E^3 f = f - (\alpha_\mu - \alpha_3)\mathfrak{g} + (\alpha_\mu - \alpha_3)^2|E_\mu\psi(-1), \widetilde{\widehat{N-1}}|_{[3]}, \tag{A.27b}$$

$$E_\mu \mathfrak{g} = |E_\mu \psi(-1), \widehat{N-2}|_{[3]} - (\alpha_\mu - \alpha_3)|E_\mu \psi(-1), -1, \widetilde{N-2}|_{[3]}, \tag{A.27c}$$

$$\frac{(-1)^N}{|\Gamma_\mu|} E^\mu f = E_3 f - (\alpha_\mu + \alpha_3)|\mathring{E}^\mu \psi(-1), \widehat{N-2}|_{[3]}, \tag{A.27d}$$

$$\frac{(-1)^N}{|\Gamma_\mu|} E^\mu E^3 f = f + (\alpha_\mu + \alpha_3)\mathfrak{g} - (\alpha_\mu + \alpha_3)^2|\mathring{E}^\mu \psi(-1), \widetilde{N-1}|_{[3]}, \tag{A.27e}$$

$$\frac{(-1)^N}{|\Gamma_\mu|} E^\mu \mathfrak{g} = -|\mathring{E}^\mu \psi(-1), \widehat{N-2}|_{[3]} - (\alpha_\mu + \alpha_3)|\mathring{E}^\mu \psi(-1), -1, \widetilde{N-2}|_{[3]}; \tag{A.27f}$$

$$\begin{aligned}(a-b)\overline{\underset{\hat{\,}}{\underset{\sim}{f}}} = {} & (a-b)\underset{\sim}{f} - (a-c)^2|\underset{\sim}{\psi}(-1), \widehat{N-2}|_{[3]} + (b-c)^2|\underset{\hat{\,}}{\psi}(-1), \widehat{N-2}|_{[3]} \\ & + (a-c)^2(b-c)|\underset{\sim}{\psi}(-1), -1, \widetilde{N-2}|_{[3]} - (a-c)(b-c)^2|\underset{\hat{\,}}{\psi}(-1), \\ & -1, \widetilde{N-2}|_{[3]} + (a-c)^2(b-c)^2|\underset{\hat{\,}}{\psi}(-1), \underset{\sim}{\psi}(-1), \widetilde{N-2}|_{[3]},\end{aligned} \tag{A.27g}$$

$$\begin{aligned}\frac{a-b}{|\Gamma_1||\Gamma_2|}\widehat{\widetilde{f}} = {} & (a-b)\underset{\sim}{f} - (a+c)^2|\mathring{E}^1 \psi(-1), \widehat{N-2}|_{[3]} + (b+c)^2|\mathring{E}^2 \psi(-1), \widehat{N-2}|_{[3]} \\ & - (a+c)^2(b+c)|\mathring{E}^1 \psi(-1), -1, \widetilde{N-2}|_{[3]} + (a+c)(b+c)^2|\mathring{E}^2 \psi(-1), \\ & -1, \widetilde{N-2}|_{[3]} - (a+c)^2(b+c)^2|\mathring{E}^2 \psi(-1), \mathring{E}^1 \psi(-1), \widetilde{N-2}|_{[3]}.\end{aligned} \tag{A.27h}$$

To prove (A.27*a*), noting that

$$\underset{\sim}{\psi}(l+1) = \psi(l) - (a-c)\underset{\sim}{\psi}(l), \tag{A.28}$$

we have

$$\begin{aligned}\underset{\sim}{f} &= |\underset{\sim}{\psi}(0), \ldots, \underset{\sim}{\psi}(N-3), \underset{\sim}{\psi}(N-2), \psi(N-2) - (a-c)\underset{\sim}{\psi}(N-2)|_{[3]} \\ &= |\underset{\sim}{\psi}(0), \ldots, \underset{\sim}{\psi}(N-3), \underset{\sim}{\psi}(N-2), \psi(N-2)|_{[3]}.\end{aligned}$$

Repeating this until we obtain

$$\underset{\sim}{f} = |\psi(-1) - (a-c)\underset{\sim}{\psi}(-1), \psi(0), \ldots, \psi(N-4), \psi(N-3), \psi(N-2)|_{[3]},$$

which is (A.27*a*) for $\mu = 1$. In a similar way we can prove (A.27*b*), (A.27*c*) and (A.27*g*).

To prove (A.27*d*), (A.27*e*), (A.27*f*) and (A.27*h*), we consider the Casoratians $f = |\widehat{N-1}|_{[3]}$, $\mathfrak{g} = |-1, \widetilde{N-1}|_{[3]}$ with entries (A.26) and denote such f and $\mathfrak{g}$ by $f_{\mathrm{IV}[3]}$ and $\mathfrak{g}_{\mathrm{IV}[3]}$. Noting that σ satisfies

$$(a+c)\widetilde{\sigma}(l) = \sigma(l) + \widetilde{\sigma}(l+1), \tag{A.29}$$

$$(b+c)\widehat{\sigma}(l) = \sigma(l) + \widehat{\sigma}(l+1), \tag{A.30}$$

$$(a-b)\widehat{\widetilde{\sigma}}(l) = \widehat{\sigma}(l) - \widetilde{\sigma}(l), \tag{A.31}$$

we can have some formulas for $E^\mu f_{\mathrm{IV}[3]}, E^\mu E^3 f_{\mathrm{IV}[3]}, E^\mu \mathfrak{g}_{\mathrm{IV}[3]}$ with $\mu = 1, 2$, etc. Then, using the relationship $f_{\mathrm{I}[3]} = |\Gamma_1|^n|\Gamma_2|^m f_{\mathrm{IV}[3]}$, $\mathfrak{g}_{\mathrm{I}[3]} = |\Gamma_1|^n|\Gamma_2|^m \mathfrak{g}_{\mathrm{IV}[3]}$ and $\psi = \Gamma_1^n \Gamma_2^m \sigma$, we can finally derive the formulas (A.27*d*), (A.27*e*), (A.27*f*) and (A.27*h*).

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